Adding independent random variables

Starter

1. **(Review of last lesson)** A model is proposed for the weight distribution of trout in a fish farm. The density function of the model is $f(x) = \begin{cases} \frac{1}{36}w(6-w) & 0 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$.

A sample of 80 fish is obtained and their weights are recorded. The table contains a summary of the data.

Weight class	$0 \le w < 1$	$1 \le w < 2$	$2 \le w < 4$	$4 \le w < 6$
Number of fish	6	14	39	21

(a) Use the model to determine the expected number of fish in each weight class.
(b) Conduct a goodness-of-fit test at the 10% level to determine if the model is appropriate for these data.

Notes

In AS level we looked at the linear combination of random variables and found that:

$$E(aX+b) = aE(X)+b \quad \text{and} \quad \text{Var}(aX+b) = a^2\text{Var}(X)$$

At A2, we include the combination of random variables

E.g. 1 From AS level we have $\text{E}(aX+b) = a\text{E}(X)+b$ and $\text{Var}(aX+b) = a^2\text{Var}(X)$.
Given that if X and Y are independent random variables then $E(X+Y) = E(X)+E(Y)$ and $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)$, conjecture expressions for the following:

(a) $E(X-Y)$
(b) $\text{Var}(X-Y)$
(c) $E(aX+bY+c) = aE(X)+bE(Y)+c$
(d) $\text{Var}(aX+bY+c) = a^2\text{Var}(X)+b^2\text{Var}(Y)$

Working: (a) $E(X-Y) = E(X+(-Y)) = E(X)-E(Y)$

If X and Y are any two random variables, then:

$$E(aX \pm bY \pm c) = aE(X) \pm bE(Y) \pm c$$

If X and Y are independent random variables, then:

$$\text{Var}(aX \pm bY \pm c) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

N.B. X and Y have to be independent but they do not have to be from different populations.

E.g. 2 Independent random variables X and Y are such that $E(X)=3$, $E(Y)=5$, $\text{Var}(X)=4$ and $\text{Var}(Y)=2$. Find:

(a) $E(4X+2Y)$
(b) $E(6X-Y)$
(c) $\text{Var}(3X+6Y+11)$
(d) $\text{Var}(5Y-3X)$
(e) $\text{Var}(3X-5Y)$

E.g. 3 Let X_1 and X_2 be independent observations of the random variable X so that:

$$E(X_1) = E(X_2) = E(X) \quad \text{and} \quad \text{Var}(X_1) = \text{Var}(X_2) = \text{Var}(X)$$

Show that

(a) $E(X_1 + X_2) = E(2X)$

(b) $\text{Var}(X_1 + X_2) \neq \text{Var}(2X)$?

Working: (a)

$$\begin{aligned} E(X_1 + X_2) &= E(X_1) + E(X_2) \\ &= E(X) + E(X) \\ &= 2E(X) \\ &= E(2X) \end{aligned}$$

In general: $E(X_1 + X_2) = E(2X)$ but $\text{Var}(X_1 + X_2) \neq \text{Var}(2X)$

$\text{Var}(2X) = 4\text{Var}(X)$

$\text{Var}(X_1 + X_2) = 2\text{Var}(X)$

where X_1 and X_2 are independent observations of the random variable X.
i.e. the expectations of combinations is the same but the variability of a single observation doubled is greater than the variability of two independent observations added together. With two independent observations there is the possibility that they will cancel each out to some extent.

E.g. 4 A crane is lifting a crate with 4 large boxes and 5 small boxes. The large boxes have mean mass 18 kg and standard deviation 3 kg while the small boxes have mean mass 12 kg and standard deviation 1.5 kg. Given that the crate has mass 25 kg, calculate the expectation and standard deviation of the total mass of the crate with 4 large boxes and 5 small boxes loaded on it.

E.g. 5 Two Year 7 tutor groups, A and B, take a maths test and the details of their results are:

	Mean	**Standard deviation**
$7A$	69	8.1
$7B$	75	5.2

One student from $7A$ and one student from $7B$ are selected at random. Calculate the expected mean and standard deviation of the difference between their scores.

Video: [Combining independent random variables]

[Solutions to Starter and E.g.s]

Exercise

p153 8A Qu 1i, 3-6, (7 red)

Summary

If X and Y are any two random variables, then:

$$E(aX \pm bY \pm c) = aE(X) \pm bE(Y) \pm c$$

If X and Y are independent random variables, then:

$$\text{Var}(aX \pm bY \pm c) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

N.B. X and Y have to be independent but they do not have to be from different populations.

$\text{Var}(2X) = 2^2\text{Var}(X) = 4\text{Var}(X)$

$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$ where X_1 and X_2 are independent observations of the random variable X.