Area enclosed by a curve

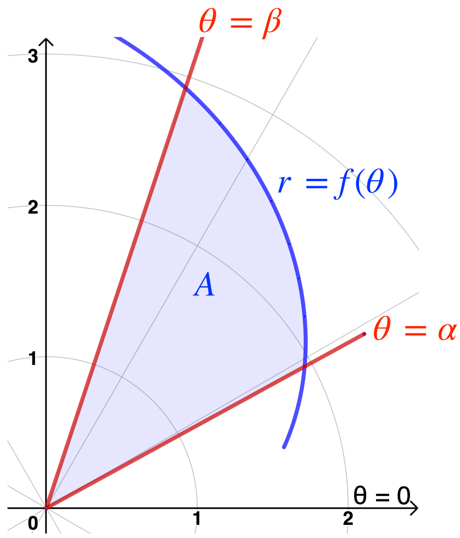
Starter

1. **(Review of previous material)** Find the exact value of $\int_0^{\frac{\pi}{4}} (1 + 2\cos\theta)^2 d\theta$.

Notes

In Cartesian coordinates, integration finds the area between the curve, ***two vertical lines*** $x = a$ and $x = b$, and the $x-$axis (or $y-$axis).

In polar coordinates, the area is found between the curve, the pole and the ***two half lines*** $\theta = \alpha$ and $\theta = \beta$, which pass through the pole.



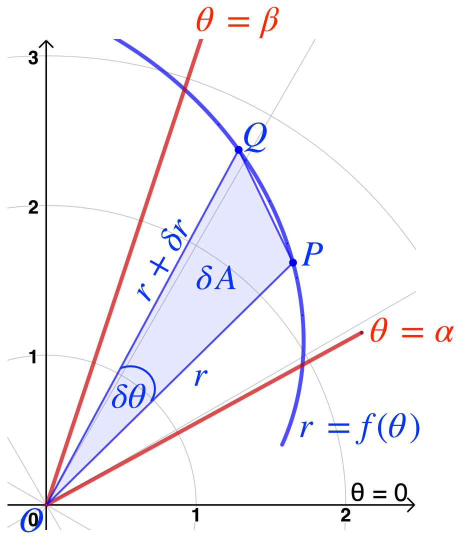
The area is given by: $$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

This is given in the formula booklet.

Proof of the area formula for polar coordinates

Let points $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two points on the curve with equation $r = f(\theta)$.

If P and Q are close together then the area δA is approximately equal to the area of the triangle OPQ.



i.e. $$\delta A \approx \frac{1}{2} r(r + \delta r)\sin\delta\theta$$

$$\Rightarrow \quad \frac{\delta A}{\delta\theta} \approx \frac{1}{2} r(r + \delta r)\frac{\sin\delta\theta}{\delta\theta}$$

As $\delta\theta \to 0$: $\delta r \to 0$, $\frac{\sin\delta}{\delta\theta} \to 1$ and $\frac{\delta A}{\delta\theta} \to \frac{dA}{d\theta}$.

$$\therefore \frac{dA}{d\theta} = \frac{1}{2} r^2$$

By summing these small areas from $\theta = \alpha$ to $\theta = \beta$: $$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

E.g. 1 A curve has equation $r = 3 + 3\sin\theta$. Find the area enclosed by the curve.

Working: Area enclosed

$$\begin{aligned}
\text{Area enclosed} &= \int_0^{2\pi} \frac{1}{2}(3 + 3\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (9 + 18\sin\theta + 9\sin^2\theta) d\theta \\
&= \frac{9}{4}\int_0^{2\pi} (2 + 4\sin\theta + 1 - \cos 2\theta) d\theta \\
&= \frac{9}{4}\left[(3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta)\right]_0^{2\pi} \\
&= \frac{9}{4}\Big((6\pi - 4 - 0) - (0 - 4 - 0)\Big) \\
&= \frac{27\pi}{2}
\end{aligned}$$

E.g. 2 A curve has equation $r = 5\cos 4\theta$. Find the area of one petal of the curve.

N.B. Make sure the curve is defined for the angles.

E.g. 3 Find the area of the limaçon $r = 1 + 2\cos\theta$.

Video: **Area bounded by a polar curve**

Solutions to Starter and E.g.s

Exercise
p211 9D Qu 1i, 2-9

Summary
The area is found between the curve $r = f(\theta)$, the pole and the two half lines $\theta = \alpha$ and $\theta = \beta$, which pass through the pole is given by:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

This is given in the formula booklet.