Averages and measures of spread of continuous random variables

Starter

1. **(Review of last lesson)**

The life, X, of the StayBrite light bulb is modelled by the probability density function

$f(x) = \begin{cases} ke^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, where X is measured in thousands of hours.

(a) Find k.

(b) Find the probability that a StayBrite bulb lasts longer than 1000 hours.

(c) Find the probability that a StayBrite bulb lasts less than 500 hours.

Notes

The expectation or mean of a continuous random variable, X, is defined by:

$$\text{E}(X) = \mu = \int_{-\infty}^{\infty} xf(x)dx$$

The variance of a continuous random variable, X, is defined by:

$$\text{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

These formulae appear in the formula booklet.

From AS level, discrete random variables used $\text{Var}(X) = E(X^2) - E^2(X)$ and this is also true for crv with $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

E.g. 1 The probability density function $f(x) = \begin{cases} 2x - 4 & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$.

Find the mean and variance.

Working: Mean $= \displaystyle\int_{-\infty}^{\infty} xf(x)dx = \int_{2}^{3} x(2x-4)dx = \frac{8}{3}$

Variance $= \displaystyle\int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

$= \displaystyle\int_{2}^{3} x^2(2x-4)dx - \left(\frac{8}{3}\right)^2$

$= \dfrac{1}{18}$

E.g. 2 The EverOn torch battery has a life of X hours . The variable X is modelled by the probability density function $f(x) = \begin{cases} 3000x^{-4} & x \geq 10 \\ 0 & \text{otherwise} \end{cases}$

Find the mean and standard deviation of EverOn torch batteries.

Mode and median of a continuous random variable

The ***mode*** of a continuous random variable is the $x-$ ***value*** that gives the probability density function its ***maximum value*** over the given domain. Make sure you check the ***end points*** as these could give a higher point than a maximum turning point. A sketch of the graph can help.

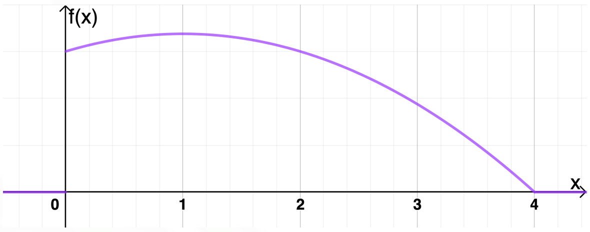
The ***median***, m, of a continuous random variable is the value such that $P(X \leq m) = \frac{1}{2}$.

i.e. $$\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$$

E.g. 3 The crv X has pdf where $f(x) = \begin{cases} \frac{3}{80}(2+x)(4-x) & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$.

Find the mode and median.

Working: The graph indicates that the mode will be the maximum value of the curve.



Given that the roots occur when $x = -2$ and $x = 4$, the maximum will occur when $x = 1$.
So the mode is 1.

Median, $\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$: $\quad \int_{0}^{m} \frac{3}{80}(2+x)(4-x)dx = \frac{1}{2}$

$\int_{0}^{m} (8 + 2x - x^2)dx = \frac{40}{3} \quad \Rightarrow \quad \left[8x + x^2 - \frac{1}{3}x^3\right]_0^m = \frac{40}{3}$

$24m + 3m^2 - m^3 = 40 \quad \Rightarrow \quad m^3 - 24m - 3m^2 + 40 = 0$

Since $0 \leq m \leq 4$, $m \approx 1.26$

The mode is 1 and the median is 1.26 (3 s.f.).

E.g. 4 A continuous random variable is modelled by $f(x) = \begin{cases} \frac{1}{6}\left(x^2 - 4x + 5\right) & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$.

Find the mode and median.

Video: **Mean and variance of continuous random variable**

Solutions to Starter and E.g.s

Exercise
p125 7B Qu 1i, 2i, 3-7 (8 red)

Summary

Mean: $\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$

Variance: $\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$ ***...or...*** $\mathrm{Var}(X) = E(X^2) - E^2(X)$

Mode: the $x-$ ***value*** that gives the pdf its ***maximum value*** over the given domain. Check the ***end points***.

Median, m: $P(X \leq m) = \frac{1}{2}$ i.e. $\int_{-\infty}^{m} f(x) dx = \frac{1}{2}$