Centres of mass by integration

Starter

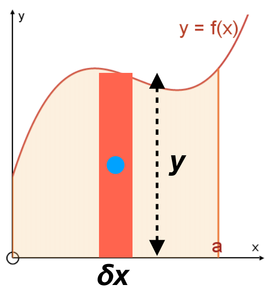
1. **(Review of last lesson)** A hat consists of a hemispherical shell of radius 10 cm with a brim of outside radius 15 cm. Assuming the material is uniform thickness, find the distance of the centre of mass from the plane of the brim.

2. **(Review of A2 FM Pure material)** Find the volume of the solid formed when the curve $y = \cos x$ is rotated 2π about the $x-$axis between $x = 0$ and $x = \dfrac{\pi}{2}$.

Notes

Centre of mass of a uniform lamina defined by a function f(x)

Let a uniform lamina be defined by $f(x)$ and the lines $x = a$, $y = 0$ and $x = 0$).

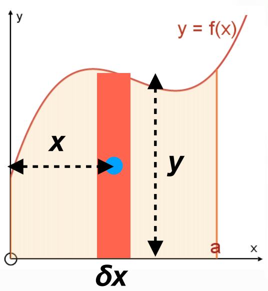


Since it is uniform, its mass is proportional to its area so we use area rather than mass. The area is given by $\displaystyle\int_0^a f(x)dx$.

The curve is split into thin rectangles of width δx and height y. The $y-$coordinate of the centre of mass of this rectangle is $\dfrac{1}{2}y$.

$x-$coordinate of the centre of mass

Take moments about the $y-$axis



Area of rectangle: $y\delta x$

Distance from the $y-$axis*: x

Moment of rectangle: $xy\delta x$

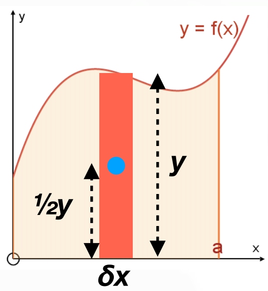
Sum of moments: $\displaystyle\int_0^a xy\,dx$

Since $y = f(x)$, the centre of mass lies at:

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx}$$

$y-$coordinate of the centre of mass

Take moments about the $x-$axis



Area of rectangle: $y\delta x$

Distance from the $x-$axis: $\dfrac{1}{2}y$

Moment of rectangle: $\dfrac{1}{2}y^2\delta x$

Sum of moments: $\displaystyle\int_0^a \frac{1}{2}y^2dx$

Since $y = f(x)$, the centre of mass lies at:

$$\bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2dx}{\int_0^a f(x)dx}$$

N.B. It could be argued that the distance from the $y-$axis* is $x+\frac{1}{2}\delta x$, but the $\frac{1}{2}\delta x$ term disappears in the limit as $\delta x \to \infty$.

In general, the centre of mass lies at:

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2 dx}{\int_0^a f(x)dx}$$

i.e. $A\bar{x} = \int_0^a xf(x)dx$ $\qquad A\bar{y} = \frac{1}{2}\int_0^a [f(x)]^2 dx$ $\qquad$ where A is the area of the lamina

N.B. The centre of mass of a uniform lamina can also be called the ***centroid***.

E.g. 1 Find the coordinates of the centroid of the region between the curve $y = 4 - x^2$ and the positive $x-$ and $y-$axis.

Working: Solving $4 - x^2 = 0$ gives $x = \pm 2$ — we need to consider only the positive part.

$$A = \int_0^2 (4-x^2)dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3}$$

$$A\bar{x} = \int_0^2 x(4-x^2)dx = \int_0^2 (4x - x^3)dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 4$$

$$A\bar{y} = \frac{1}{2}\int_0^2 (4-x^2)^2 dx$$

$$= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)dx$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$

$$= \frac{128}{15}$$

The coordinates of the centroid are $(0.75,\ 1.6)$

E.g. 2 A uniform lamina occupies the region bounded by the $x-$axis, the line $x = 2$ and the curve $y = 3x^2$ for $0 \le x \le 2$. Find the coordinates of the centre of mass of this lamina.

Centre of mass of a uniform solid of revolution

When a uniform solid is rotated about the $x-$axis, the $y-$coordinate of its centre of mass will be zero i.e. it will lie of the $x-$axis. Therefore, we just need to find the $x-$coordinate.

Similar to the uniform lamina defined by a function, rather than mass we use the volume of the solid since mass is proportional to volume i.e. $\int_0^a \pi y^2 dx$

For a uniform solid of revolution with radius defined by $f(x)$, the the $x-$coordinate of the centre of mass is at $\bar{x} = \dfrac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx}$.

Given that π will always cancel, we could simplify this to: $\bar{x} = \dfrac{\int_0^a x y^2 dx}{\int_0^a y^2 dx}$

In general: $V\bar{x} = \displaystyle\int_0^a \pi x y^2 dx$ where V is the volume of the solid

E.g. 3 A machine component has the form of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$ for which $x \geq 0$ about the $x-$axis, the units being cm. Find the position of the centre of mass.

Working:

$$\int_0^9 \pi x y^2 dx = \pi \int_0^9 x(9-x)dx$$
$$= \pi \int_0^9 (9x - x^2)dx$$
$$= \pi \left[\frac{9}{2}x^2 - \frac{1}{3}x^3\right]_0^9$$
$$= 121.5\pi$$

Volume $= \displaystyle\int_0^9 \pi(9-x)dx = \pi\left[(9x - \frac{1}{2}x^2)\right]_0^9 = 40.5\pi$

So the $x-$coordinate of the centre of mass is $\dfrac{121.5\pi}{40.5\pi} = 3$ cm from the origin.

E.g. 4 The spike of a swordfish is modelled as the solid obtained by rotating the curve $y = 1 + 0.001x^{\frac{3}{2}}$ about the $x-$axis for $0 \leq x \leq 100$, the units being cm. How far from the end is the centre of mass of the spike?

Video (password needed): Centre of mass of a lamina by integration
Video: Centre of mass by integration
Video: Centre of mass (solid of revolution)

Solutions to Starter and E.g.s

Exercise
p268 10A Qu 1-10

Summary
Uniform lamina (defined by $f(x)$ and the lines $x = a$, $y = 0$ and $x = 0$) centre of mass at:

$$\bar{x} = \frac{\int_0^a x f(x) dx}{\int_0^a f(x) dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2 dx}{\int_0^a f(x) dx}$$

Uniform solid of revolution with radius defined by $f(x)$ is:

$$\bar{x} = \frac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx} \qquad \text{where } \int_0^a \pi y^2 dx \text{ is the volume of the solid of revolution}$$