Centres of mass by integration

Starter

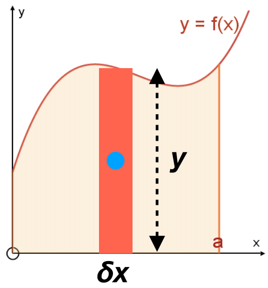
1. (Review of last lesson) A hat consists of a hemispherical shell of radius 10 cm with a brim of outside radius 15 cm. Assuming the material is uniform thickness, find the distance of the centre of mass from the plane of the brim.

2. (Review of A2 FM Pure material) Find the volume of the solid formed when the curve $y = \cos x$ is rotated 2π about the $x-$axis between $x = 0$ and $x = \dfrac{\pi}{2}$.

Notes

Centre of mass of a uniform lamina defined by a function f(x)

Let a uniform lamina be defined by $f(x)$ and the lines $x = a$, $y = 0$ and $x = 0$).

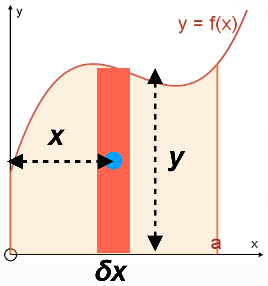


Since it is uniform, its mass is proportional to its area so we use area rather than mass. The area is given by $\displaystyle\int_0^a f(x)dx$.

The curve is split into thin rectangles of width δx and height y. The $y-$coordinate of the centre of mass of this rectangle is $\dfrac{1}{2}y$.

$x-$coordinate of the centre of mass

Take moments about the $y-$axis



Area of rectangle: $y\delta x$

Distance from the $y-$axis*: x

Moment of rectangle: $xy\delta x$

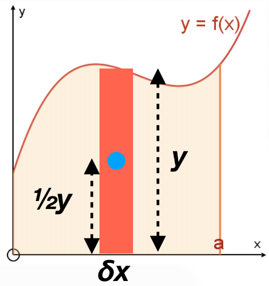
Sum of moments: $\displaystyle\int_0^a xy\,dx$

Since $y = f(x)$, the centre of mass lies at:

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx}$$

$y-$coordinate of the centre of mass

Take moments about the $y-$axis



Area of rectangle: $y\delta x$

Distance from the $y-$axis: $\dfrac{1}{2}y$

Moment of rectangle: $\dfrac{1}{2}y^2\delta x$

Sum of moments: $\displaystyle\int_0^a \frac{1}{2}y^2dx$

Since $y = f(x)$, the centre of mass lies at:

$$\bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2dx}{\int_0^a f(x)dx}$$

N.B. It could be argued that the distance from the $y-$axis* is $x + \frac{1}{2}\delta x$, but the $\frac{1}{2}\delta x$ term disappears in the limit as $\delta x \to \infty$.

In general, the centre of mass lies at:

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2 dx}{\int_0^a f(x)dx}$$

i.e. $A\bar{x} = \int_0^a xf(x)dx$ $\qquad A\bar{y} = \frac{1}{2}\int_0^a [f(x)]^2 dx$ $\qquad$ where A is the area of the lamina

N.B. The centre of mass of a uniform lamina can also be called the ***centroid***.

E.g. 1 Find the coordinates of the centroid of the region between the curve $y = 4 - x^2$ and the positive $x-$ and $y-$axis.

Working: Solving $4 - x^2 = 0$ gives $x = \pm 2$ — we need to consider only the positive part.

$$A = \int_0^2 (4 - x^2)dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3}$$

$$A\bar{x} = \int_0^2 x(4 - x^2)dx = \int_0^2 (4x - x^3)dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 4$$

$$A\bar{y} = \frac{1}{2}\int_0^2 (4 - x^2)^2 dx$$

$$= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)dx$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$

$$= \frac{128}{15}$$

The coordinates of the centroid are $(0.75,\ 1.6)$

E.g. 2 A uniform lamina occupies the region bounded by the $x-$axis, the line $x = 2$ and the curve $y = 3x^2$ for $0 \le x \le 2$. Find the coordinates of the centre of mass of this lamina.

Centre of mass of a uniform solid of revolution

When a uniform solid is rotated about the $x-$axis, the $y-$coordinate of its centre of mass will be zero i.e. it will lie of the $x-$axis. Therefore, we just need to find the $x-$coordinate.

Similar to the uniform lamina defined by a function, rather than mass we use the volume of the solid since mass is proportional to volume i.e. $\int_0^a \pi y^2 dx$

For a uniform solid of revolution with radius defined by $f(x)$, the the $x-$coordinate of the centre of mass is at $\bar{x} = \dfrac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx}$.

Given that π will always cancel, we could simplify this to: $\bar{x} = \dfrac{\int_0^a x y^2 dx}{\int_0^a y^2 dx}$

In general: $V\bar{x} = \displaystyle\int_0^a \pi x y^2 dx$ where V is the volume of the solid

E.g. 3 A machine component has the form of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$ for which $x \geq 0$ about the $x-$axis, the units being cm. Find the position of the centre of mass.

Working:

$$\begin{aligned}\int_0^9 \pi x y^2 dx &= \pi \int_0^9 x(9-x)dx \\ &= \pi \int_0^9 (9x - x^2)dx \\ &= \pi \left[\frac{9}{2}x^2 - \frac{1}{3}x^3\right]_0^9 \\ &= 121.5\pi\end{aligned}$$

$$\text{Volume} = \int_0^9 \pi(9-x)dx = \pi\left[(9x - \frac{1}{2}x^2)\right]_0^9 = 40.5\pi$$

So the $x-$coordinate of the centre of mass is $\dfrac{121.5\pi}{40.5\pi} = 3$ cm from the origin.

E.g. 4 The spike of a swordfish is modelled as the solid obtained by rotating the curve $y = 1 + 0.001x^{\frac{3}{2}}$ about the $x-$axis for $0 \leq x \leq 100$, the units being cm. How far from the end is the centre of mass of the spike?

Video (password needed): Centre of mass of a lamina by integration
Video: Centre of mass by integration
Video: Centre of mass (solid of revolution)

Solutions to Starter and E.g.s

Exercise
p268 10A Qu 1-10

Summary
Uniform lamina (defined by $f(x)$ and the lines $x = a$, $y = 0$ and $x = 0$) centre of mass at:

$$\bar{x} = \frac{\int_0^a x f(x) dx}{\int_0^a f(x) dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^a [f(x)]^2 dx}{\int_0^a f(x) dx}$$

Uniform solid of revolution with radius defined by $f(x)$ is:

$$\bar{x} = \frac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx}$$ where $\int_0^a \pi y^2 dx$ is the volume of the solid of revolution