Complex exponents (including Euler's formula)

Starter

1. **(Review of last lesson)** Show that $(-1+\text{i})^{16}$ is real and that $\dfrac{1}{(-1+\text{i})^6}$ is purely imaginary, giving the value of each.

2. The standard Maclaurin series for e^x is $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\ldots+\dfrac{x^r}{r!}+\ldots$.
   (a) Find the Maclaurin series for $e^{\text{i}\theta}$.
   (b) By considering the Maclaurin series for $\cos\theta$ and $\sin\theta$, express $e^{\text{i}\theta}$ in terms of $\cos\theta$ and $\sin\theta$.

Notes

So $re^{\text{i}\theta} = r(\cos\theta + \text{i}\sin\theta)$ where $r > 0$ and $-\pi < \theta \le \pi$ or $0 \le \theta < 2\pi$.

E.g. 1 State the value of $e^{\text{i}\pi}$ — this is known as Euler's formula.

Working: $e^{\text{i}\pi} = \cos\pi + \text{i}\sin\pi = -1$
i.e. $\text{irrational number}^{\text{imaginary number} \times \text{irrational number}} = -1$

Useful results

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$
$\sin\theta = \cos\left(\theta - \dfrac{\pi}{2}\right) = \cos\left(\dfrac{\pi}{2} - \theta\right)$
$\cos\theta = \sin\left(\theta + \dfrac{\pi}{2}\right)$

E.g. 2 Express each of the complex number in the form $re^{i\theta}$:

(a) $1+\text{i}$ (b) $\sqrt{3}-\text{i}$
(c) $\dfrac{1+\text{i}}{\sqrt{3}-\text{i}}$ (d) $(1+\text{i})(\sqrt{3}-\text{i})$

Working: (a) $1+\text{i} = \sqrt{2}e^{\text{i}\frac{\pi}{4}}$

E.g. 3 Express in the form a + ib:

(a) $4e^{\text{i}\frac{\pi}{3}}$ (b) $5e^{\text{i}\pi}$ (c) $e^{-\text{i}\frac{\pi}{2}}$

Working: (a) $4e^{\text{i}\frac{\pi}{3}} = 4\left(\cos\dfrac{\pi}{3} + \text{i}\sin\dfrac{\pi}{3}\right) = 4\left(\dfrac{1}{2} + \text{i}\dfrac{\sqrt{3}}{2}\right) = 2 + \text{i}2\sqrt{3}$

E.g. 4 Find expressions for $e^{i\theta} + e^{-i\theta}$ and $e^{i\theta} - e^{-i\theta}$ in terms of $\cos\theta$ and/or $\sin\theta$

Working:
$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + \text{i}\sin\theta + \cos(-\theta) + \text{i}\sin(-\theta) \\ &= \cos\theta + \text{i}\sin\theta + \cos\theta - \text{i}\sin\theta \\ &= 2\cos\theta \end{aligned}$$

Exponential raised to the power of a complex number

E.g. 5 Given that $z = 2 + 3\text{i}$, express e^z in the form $a + b\text{i}$, giving your answer to 4 s.f..

Video: De Moivre's Theorem

Video: Exponential form (Euler's relation)

Solutions to Starter and E.g.s

Exercise

p33 2B Qu 1i, 2i, 3i, 4i, 5-12

Summary

$re^{\text{i}\theta} = r(\cos\theta + \text{i}\sin\theta)$

Euler's formula: $e^{\text{i}\pi} = -1$

If $z = x + \text{i}y$, the definition of e^z is $e^z = e^{x+\text{i}y} = e^x \times e^{\text{i}y} = e^x(\cos y + \text{i}\sin y)$.