Confidence intervals

Starter

1. **(Review of last lesson)**
A fish species in Wales is normally distributed with mean body length of 88.2 mm and variance 9 mm. Researchers believe the same species in Scotland is larger. A random sample of 40 taken from Scotland gave the values $\sum x_i = 3569.4$ and $\sum x_i^2 = 318935.9$. Test at the 5% level, using the variance value calculated from the data of the random sample, whether the fish off the coast of Scotland are longer.

Notes

From a sample, the value of the population mean, μ, is estimated as $\bar{x}$. It would be useful to give a range of values between which μ lies and to provide some measure of how reliable this range of values is.

Imagine a friend phoned last week but you can't exactly remember when it was. Without consulting the recent calls on your phone, you could possibly make these statements:

"I am 5% certain it was at exactly $8:30$ pm."
"I am 80% certain it was between $8:15$ pm and $8:45$ pm."
"I am 95% certain it was between $8:00$ pm and $9:00$ pm."
"I am 100% certain it was on Wednesday."

Notice how the width of the interval increases as the confidence about the statement increases.

In statistics, these intervals are called ***confidence intervals***.

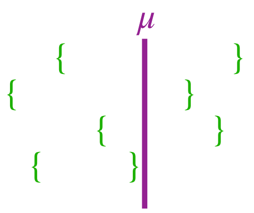
The ***wider*** the confidence interval, the ***greater the certainty***.
The ***narrower*** the confidence interval, the ***smaller the certainty***.

Important points

1. μ ***is fixed***, but unknown, and therefore does not have a distribution. It makes no sense to say that μ is between certain values.
2. Since μ is fixed, it is the ***confidence interval that varies*** according to the values of $\bar{x}$ and the size of the sample, n.
3. A 95% confidence interval tells us that the probability that the interval contains μ is 0.95.

Suppose 50 samples were taken and 95% confidence intervals for μ were calculated for each sample. This would give 50 different confidence intervals, each centred on the 50 different values for $\bar{x}$. We would expect 95% of the confidence intervals to contain μ.

Let the curly brackets represent confidence intervals (CI) from different samples. The centre of the confidence intervals is the value of $\bar{x}$.



μ is fixed
1. The confidence interval contains μ.
2. Same width as 1st CI so n and level of confidence are the same.
3. Narrower CI — either n is higher or confidence level is lower.
4. The confidence interval does not contain μ.

Each sample produces a different mean and so the confidence interval moves accordingly.

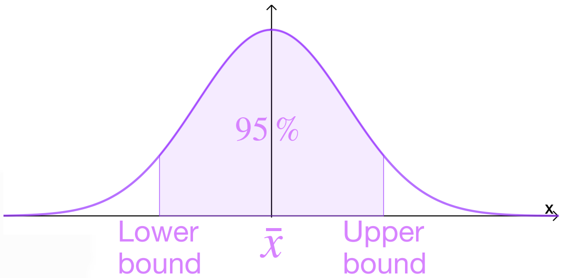
The ***width of the confidence interval*** is based on the ***size of sample***, n, the ***variance of the population*** and the ***level of confidence***.

Calculating confidence intervals

Consider a random variable X which is normally distributed such that $X \sim \mathrm{N}(\mu, \sigma^2)$. Hence, the sample mean satisfies $\bar{X}_n \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$, where μ is the population mean.

To find the $95\,\%$ confidence interval, we centre the interval on the sample mean, $\bar{x}$, and find the lower and upper bounds such that $95\,\%$ of the area under the normal curve is symmetrically distributed about $\bar{x}$.

i.e. $\quad P(\bar{x} - a < \bar{X} < \bar{x} + a) = 0.95$



E.g. 1 (a) Given that $P(\bar{x} - a < \bar{X} < \bar{x} + a) = 0.95$, find the $Z-$ value that corresponds to a i.e $P(-z < Z < z) = 0.95$. Give your answer to 3 s.f..

(b) Using the formula $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$ and the value found in (a), rearrange to find an inequality of the form $-k < \mu < k$. This is the $95\,\%$ confidence interval.

(c) Find similar inequalities of the form $-k < \mu < k$ for a:

(i) $90\,\%$ confidence interval

(ii) $98\,\%$ confidence interval

(d) State the width of the $98\,\%$ confidence interval.

Working: (a) $P(-z < Z < z) = 0.95 \quad \Rightarrow \quad P(Z < z) = 0.975$

$z = 1.96$ (3 s.f.)

(b) $P(-1.96 < Z < 1.96) = 0.95$

Let $z = 1.96$

Using $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$: $\qquad 1.96 = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$$1.96 \times \frac{\sigma}{\sqrt{n}} = \bar{x} - \mu$$

$$\mu = \bar{x} - 1.96 \times \frac{\sigma}{\sqrt{n}}$$

When $z = -1.96$, $\quad \mu = \bar{x} + 1.96 \times \dfrac{\sigma}{\sqrt{n}}$

$95\,\%$ CI: $\quad \bar{x} - 1.96 \times \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \times \dfrac{\sigma}{\sqrt{n}}$

In general the width of a confidence interval is $2z \times \dfrac{\sigma}{\sqrt{n}}$ where z is the relevant $Z-$value.

E.g. 2 The number of hours for which a certain light bulb stays lit is normally distributed with a standard deviation 45 hours. A random sample of 60 lightbulbs is tested and the sample mean is 734 hours. Calculate these confidence intervals for μ.

(a) $90\,\%$ (b) $98\,\%$

Working: (a) $\bar{x} = 734,\ \sigma = 45$

$90\,\%$ CI: $$\bar{x} - 1.64 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.64 \times \frac{\sigma}{\sqrt{n}}$$

$$734 - 1.64 \times \frac{45}{\sqrt{60}} < \mu < 734 + 1.64 \times \frac{45}{\sqrt{60}}$$

The $90\,\%$ confidence interval for this sample is $724.5 < \mu < 743.5$.

E.g. 3 The $95\,\%$ confidence interval for a mean of μ is 85.3 ± 2.35. Find the following confidence intervals for μ.

(a) $90\,\%$ (b) $99\,\%$

Working: (a) $\bar{x} = 85.3$

$95\,\%$ CI: $$\bar{x} - 1.96 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \times \frac{\sigma}{\sqrt{n}}$$

So $$1.96\frac{\sigma}{\sqrt{n}} = 2.35$$

$$\frac{\sigma}{\sqrt{n}} = \frac{1.96}{2.35} \approx 0.834$$

$90\,\%$ CI: $$\bar{x} - 1.64 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.64 \times \frac{\sigma}{\sqrt{n}}$$

$$85.3 - 1.64 \times \frac{196}{235} < \mu < 85.3 + 1.64 \times \frac{196}{235}$$

The $90\,\%$ confidence interval is $83.33 < \mu < 82.27$.

E.g. 4 A normal distribution has standard deviation 8. Estimate the smallest sample size required if these confidence levels should have a width of less than 1.5.

(a) $95\,\%$ (b) $90\,\%$

Working: (a) $95\,\%$ CI: $$\bar{x} - 1.96 \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \times \frac{\sigma}{\sqrt{n}}$$

Width of less than 1.5: $$2 \times 1.96 \times \frac{8}{\sqrt{n}} < 1.5$$

$$\sqrt{n} > 2 \times 1.96 \times \frac{8}{1.5}$$

$$n > 437.1$$

The smallest sample size is 438.

What if the distribution of the population is not known?

When the distribution of the population is not known, the central limit theorem can be invoked so long as the sample size is large enough.

E.g. 5 An exam board knows that each year the standard deviation of the marks in a certain subject is 13.5 but the mean mark, μ, will fluctuate. An examiner wishes to estimate the mean mark of all candidates nationally before all results are in and takes a random sample of 250 results. The sample mean is 68.4.

(a) Calculate a $95\ \%$ confidence interval for μ.

(b) Once all results were is, the actual value of μ was 65.4. What conclusions might the examiner draw about the sample?

What if the variance of the population is not known?

It seems strange to have the variance of a population but not its mean. In real life, on many occasions neither will be known and in such cases the population variance is estimated using:

$$s^2 = \frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$

E.g. 6 A random sample of 80 chickens is weighed and it is found that $\sum x_i = 136.85$ kg and $\sum x_i^2 = 264.97$ kg², where x_i are the weights of the chickens in the sample. Find the $95\ \%$ confidence interval for μ.

Using confidence intervals with hypothesis tests

A confidence interval can be used to decide whether the null hypothesis should be rejected or not, provided that the level of significance of the test and the confidence interval are consistent.

When the population mean lies outside the confidence interval, the null hypothesis is rejected.

E.g. 7 State the percentage of the required confidence level for these hypothesis tests.

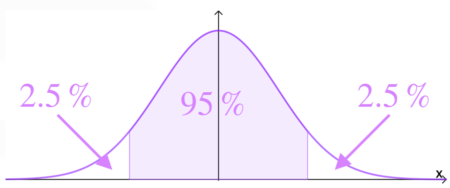
(a) A two-tailed test at the $5\ \%$ significance level.

(b) A one-tailed test at the $2\ \%$ significance level.

(c) A two-tailed test at the $\alpha\ \%$ significance level.

(d) A one-tailed test at the $\alpha\ \%$ significance level.

Working: (a) Two-tailed tests split the significant level percentage equally either side of the mean (see diagram). So a $95\ \%$ confidence interval would be needed



In general:

One-tailed test: an $\alpha\ \%$ significance level requires the $(100\% - 2\alpha\%)$ confidence interval

Two-tailed test: an $\alpha\ \%$ significance level requires the $(100\% - \alpha\%)$ confidence interval

E.g. 8 A tobacco company claims that the tar content of its cigarettes is 18.72 mg. Anti-smoking campaigners employ a research company as they think the real value is much higher. The tar content of 200 cigarettes is measure and the results, $\sum t_i = 3928.649$ mg and $\sum t_i^2 = 82415.74$ mg², where t_i is the tar content of the cigarettes. By finding a suitable confidence interval, test the company's claim at the $2\ \%$ level.

E.g. 9 The managing director of a firm commissioned a survey to estimate the mean expenditure on electrical appliances. A random sample of 100 people were questioned and the research team presented the managing director with a $95\ \%$ confidence interval of $(£128.14,\ £141.86)$. The director says this interval is too wide and wants a confidence interval of total width $£10$.

(a) Using the same value of $\bar{x}$, find the confident limits in this case.

(b) Find the level of confidence for the interval in part (a)

After reflection, the director is still not happy and now wishes to know how large a sample would be required to obtain a $95\ \%$ confidence interval of total width no more than $£10$.

(c) Find the smallest size of sample that will satisfy this request.

Video: Understanding confidence intervals
Video: How to calculate a confidence interval
Video: Confidence intervals example

Solutions to Starter and E.g.s

Exercise
p176 9B Qu 1, 2i, 3i, 4-9, (10-12 red)

Summary
The ***wider*** the confidence interval, the ***greater the certainty***.
The ***narrower*** the confidence interval, the ***smaller the certainty***.

$90\ \%$ CI: $\bar{x} - 1.64 \times \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.64 \times \dfrac{\sigma}{\sqrt{n}}$

$95\ \%$ CI: $\bar{x} - 1.96 \times \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \times \dfrac{\sigma}{\sqrt{n}}$

$98\ \%$ CI: $\bar{x} - 2.33 \times \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.33 \times \dfrac{\sigma}{\sqrt{n}}$

The number in front of $\dfrac{\sigma}{\sqrt{n}}$

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3. A $95\ \%$ confidence interval tells us that the probability that the interval contains μ is 0.95.

Using a calculator with the Normal distribution:

Probabilities Menu >> 7:Distribution >> 2:Normal CD

Input screen **Normal CD**

Lower: **type a large number like -99999 if $P(X < x)$**

Upper: **type a large number like 99999 if $P(X > x)$**

σ: **square root to get standard deviation if $X \sim$ N$(5, 7)$**

μ: ***you may need to scroll down to see this***

$x-$ ***values*** Menu >> 7:Distribution >> 3:Inverse Normal

Input screen **Inverse normal**

Area: **type a large number between 0 and 1**

σ: **square root to get standard deviation if $X \sim$ N$(5, 7)$**

μ: