Coupled first order differential equations

Starter

1. (Review of last lesson) Consider the differential equation $3\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4x = 0$.

   State the value(s) of k which would be best for these situations.
   (a) The shock absorbers on a car
   (b) A guitar string.
   (c) A soft-close toilet seat.

2. Consider the simultaneous equations:

$$\frac{dx}{dt} = 2x + 4y \qquad (1)$$
$$\frac{dy}{dt} = x - y \qquad (2)$$

   where x and y are both functions of t.
   (a) To start to solve the equations, one of the unknowns needs to be expressed in terms of the other. By choosing a suitable equation, express x as a function of y and its derivative.
   (b) By differentiating your equation from (a), find an expression for $\frac{dx}{dt}$ in terms of derivatives of y.
   (c) Substitute your equations from (a) and (b) into either equation (1) or equation (2) to form a second order differential equation in terms of y.
   (d) Solve the second order differential equation from (c) to find a general solution for y. Hence find a general solution for x.
   (e) Given that when $t = 0$, $x = 2$ and $y = -2$, find the particular solutions for x and y.
   (f) Sketch graphs of x against t, y against t and y against x. Hence state what happens as time goes on.

Notes

When two or more differential equations involve the same combination of variables they are called a ***system of differential equations***. When both equations are linear, it is a ***linear system***.

The example from the starter has two differential involving the dependent variables x and y and the independent variable t.

Success Criteria - solving coupled first order linear differential equations

As with normal simultaneous equations the key is to ***eliminate one of the dependent variables*** so that we end up with a ***second order differential equation involving just one dependent variable***:

1. ***Rearrange one equation*** to get one dependent variable in terms of the other and its derivatives.
2. ***Differentiate*** this equation.
3. Substitute into the other equation to ***eliminate one dependent variable*** and rearrange to form a ***second order differential equation*** in one dependent variable.
4. ***Solve*** the second order differential equation to find the general solution for one dependent variable.
5. ***Substitute the solution*** from step 4 Into the rearranged equation from step 1 to find the general solution for the other dependent variable.
6. Use the ***boundary conditions*** to find the constants and hence the particular solutions.

E.g. 1 A system of differential equations is given by:

$$\frac{dx}{dt} = -x + y - 1 \qquad \frac{dy}{dt} = -x - y + 3$$

with initial conditions $x = 0$ and $y = 3$ when $t = 0$.

(a) Find expressions for x and y in terms of t.

(b) Describe what happens as $t \to \infty$.

Video 1: Systems of differential equations
Video 2: Systems of differential equations

Solutions to Starter and E.g.s

Exercise
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Summary
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1. ***Rearrange one equation*** to get one dependent variable in terms of the other and its derivatives.
2. ***Differentiate*** this equation.
3. Substitute into the other equation to ***eliminate one dependent variable*** and rearrange to form a ***second order differential equation*** in one dependent variable.
4. ***Solve*** the second order differential equation to find the general solution for one dependent variable.
5. ***Substitute the solution*** from step 4 Into the rearranged equation from step 1 to find the general solution for the other dependent variable.
6. Use the ***boundary conditions*** to find the constants and hence the particular solutions.