Differential equations involving complex numbers

Starter

1. **(Review of last lesson)**
   Find the complementary function of the second order differential equations:
   (a) $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 15 = \tan x$ (b) $49\dfrac{d^2y}{dx^2} - 56\dfrac{dy}{dx} + 16y = 2$

2. Find the complementary function of the differential equation $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 65 = \sin x$.

3. **(Review of previous material)**
   Let $z = x + \mathrm{i}y$. Find: (a) $z + z^*$ (b) $z - z^*$ (c) $\mathrm{i}(z - z^*)$

Notes

In question 2 of the starter the complementary function included terms such as $e^{(\alpha+\mathrm{i}\beta)x}$.

Using basic laws of indices: $e^{(\alpha+\mathrm{i}\beta)x} = e^{\alpha x} \times e^{\mathrm{i}\beta x}$ but what does $e^{\mathrm{i}\beta x}$ equal?

E.g. 1 The standard Maclaurin series for e^x is $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots + \dfrac{x^r}{r!} + \ldots$.

(a) Find the Maclaurin series for $e^{\mathrm{i}\theta}$.

(b) By considering the Maclaurin series for $\cos\theta$ and $\sin\theta$, express $e^{\mathrm{i}\theta}$ in terms of $\cos\theta$ and $\sin\theta$.

(c) Hence express $y = Ae^{(4+7\mathrm{i})x} + Be^{(4-7\mathrm{i})x}$ in a factorised form.

E.g. 2 If $z = 2 + 3\mathrm{i}$, write down an expression in terms of cosine and sine for e^z.

E.g. 3 (a) Express $y = Ae^{(4+7\text{i})x} + Be^{(4-7\text{i})x}$ in a factorised form.

(b) From question 2 of the starter, the complementary function was $y = Ae^{(4+7\text{i})x} + Be^{(4-7\text{i})x}$. Given that $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$, find the values of A and B. What do you notice about their values?

Hint: Use $\frac{d\left(e^{(\alpha+\text{i}\beta)x}\right)}{dx} = (\alpha + \text{i}\beta)e^{(\alpha+\text{i}\beta)x}$

Working: (a)

$$\begin{aligned} y &= Ae^{(4+7\text{i})x} + Be^{(4-7\text{i})x} \\ &= Ae^{4x} \times e^{\text{i}7x} + Be^{4x} \times e^{-\text{i}7x} \\ &= e^{4x}\left(A\cos 7x + \text{i}\sin 7x + B\cos(-7x) + \text{i}\sin(-7x)\right) \\ &= e^{4x}\left(A\cos 7x + \text{i}A\sin 7x + B\cos 7x - \text{i}B\sin 7x\right) \\ &= e^{4x}\left((A+B)\cos 7x + \text{i}(A-B)\sin 7x\right) \end{aligned}$$

(b) When $x = 0, y = 1$: $A + B = 1$

$$\frac{dy}{dx} = A(4+7\text{i})e^{(4+7\text{i})x} + B(4-7\text{i})e^{(4-7\text{i})x}$$

When $x = 0$, $\frac{dy}{dx} = 1$: $A(4+7\text{i}) + B(4-7\text{i}) = 1$

$B = 1 - A$: $A(4+7\text{i}) + (1-A)(4-7\text{i}) = 1$

$$14A\text{i} + 4 - 7\text{i} = 1$$

$$A = \frac{-3+7\text{i}}{14\text{i}} = \frac{7+3\text{i}}{14}$$

$B = 1 - A$: $B = 1 - \frac{7+3\text{i}}{14} = \frac{7-3\text{i}}{14}$

The values of A and B are complex conjugate pairs.
Hence, $A + B$ and i$(A - B)$ are both real.

Proof when $b^2 - 4ac < 0$

Let the complex roots of the auxiliary equation be $\lambda = \alpha \pm \text{i}b$.

The complementary function is $y = Ae^{(\alpha+\text{i}\beta)x} + Be^{(\alpha-\text{i}\beta)x}$.

Since $e^{\text{i}\beta} = \cos\beta x + \text{i}\sin\beta x$:

$$e^{(\alpha+\text{i}b)x} = e^{\alpha x} \times e^{\text{i}bx} = e^{\alpha x}(\cos\beta x + \text{i}\sin\beta x)$$

and $e^{(\alpha-\text{i}b)x} = e^{\alpha x} \times e^{-\text{i}bx} = e^{\alpha x}\left(\cos(-\beta x) + \text{i}\sin(-\beta x)\right) = e^{\alpha x}(\cos\beta x - \text{i}\sin\beta x)$

The complementary function is

$$\begin{aligned} y &= Ae^{\alpha x}(\cos\beta x + \text{i}\sin\beta x) + Be^{\alpha x}(\cos\beta x - \text{i}\sin\beta x) \\ &= (A+B)e^{\alpha x}\cos\beta x + \text{i}(A-B)e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}\left((A+B)\cos\beta x + \text{i}(A-B)\sin\beta x\right) \end{aligned}$$

Since A and B are complex conjugate pairs, $A + B$ and i$(A - B)$ are both real.

Hence $y = e^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right)$

E.g. 4 Find the complementary function for the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13 = 4x$.

For the second order differential equation with auxiliary equation $a\lambda^2 + b\lambda + c = 0$:

Discriminant	Root(s)	Complementary function
$b^2 - 4ac > 0$	α and β	$y = Ae^{\alpha x} + Be^{\beta x}$
$b^2 - 4ac = 0$	α (repeated)	$y = (Cx + D)e^{\alpha x}$
$b^2 - 4ac = 0$	$\alpha \pm$ ib	$y = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$

E.g. 5 Find the complementary function for the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5 = \cosh x$.

Video: Homogenous linear second order differential equations

Solutions to Starter and E.g.s

Exercise
p232 10C Qu 2, 6

Summary
$e^{i\theta} = \cos\theta + i\sin\theta$ and $re^{i\theta} = r(\cos\theta + i\sin\theta)$

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