Differential Equations with Acceleration, Velocity and Displacement

Starter

1. **(Review of last lesson)** A particle moves in the direction of the vector $x\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$. The force $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is the only force acting on the particle. The speed of the particle remains constant. Find the value of x.

2. **(Review of previous material)** A curve for which $\frac{2y}{3}\frac{dy}{dx} = e^{-3x}$ has $y = 2$ when $x = 1$. Find the coordinates of the point when it crosses the y-axis. Give your answer to 4 s.f.

3. **(Review of previous material)**
Solve the differential equation $x\frac{dv}{dx} + v = x^3$ given that $v = 1$ when $x = 1$.

Notes

Acceleration can be a function of time or of displacement and often we must choose the appropriate version before setting up and solving differential equations.

Velocity as a function of displacement

When velocity is a function of time, $v = v(t)$, then $a = \frac{dv}{dt}$.

However, for an object moving in a straight line, velocity could also be a function of displacement i.e. $v = v(x)$.

In such cases, the chain rule is used:

$$\frac{dv}{dx} = \frac{dv}{dt} \div \frac{dx}{dt}$$

But $\frac{dv}{dt} = a$ and $\frac{dx}{dt} = v$:

$$\frac{dv}{dx} = \frac{a}{v}$$

$$a = v\frac{dv}{dx}$$

E.g. 1 A particle moves along a straight line in such a way that the velocity when it has travelled a distance x is given by $v = \frac{1}{p + qx}$, where p and q are constants. Find expressions for the acceleration in terms of:

(a) x

(b) v.

Working:

(a)

$$v = \frac{1}{p + qx} = (p + qx)^{-1}$$

$$\Rightarrow \frac{dv}{dx} = -q(p + qx)^{-2} = -\frac{q}{(p + qx)^2}$$

$$a = v\frac{dv}{dx} = \frac{1}{p + qx} \times -\frac{q}{(p + qx)^2}$$

$$\therefore a = -\frac{q}{(p + qx)^3}$$

(b)

$$\frac{dv}{dx} = -\frac{q}{(p + qx)^2} = -qv^2$$

So $a = -qv^3$

E.g. 2 A particle of mass 5 kg is projected along a smooth horizontal tube with a speed of 250 m/s. When it is moving at a speed of v m/s, the air resistance slowing it down is $\frac{1}{500}v^2$ N. Find an expression for the speed of the particle after it has travelled x metres.

An equation for time

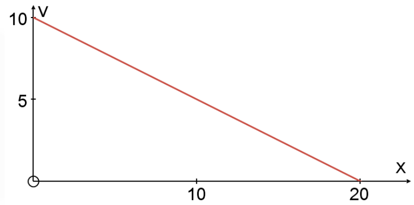
If velocity is a function of displacement, x, then re-write v as $\frac{dx}{dt}$ and solve the differential equation.

$$v(x) = \frac{dx}{dt}$$

By separating the variables find an expression for t.

$$\int 1 dt = \int \frac{1}{v(x)} dx$$
$$t = \int \frac{1}{v(x)} dx$$

E.g. 3 A car is travelling at 10 m/s when the driver applies the brakes and brings the car to rest in 20 m. The velocity reduces at a constant rate with respect to its displacement. Find an expression for the distance the car has travelled t seconds after the brakes are applied. In addition, find an expression for v in terms of t.
Hint: draw a graph of the motion in order to get a linear equation involving x and v.



Working: From the graph, we get $v = -\frac{1}{2}x + 10$

Replacing v by $\frac{dx}{dt}$: $\frac{dx}{dt} = \frac{1}{2}(20 - x)$

$$2\int \frac{1}{20 - x} dx = \int dt$$

$-2\ln(20 - x) = t + c$

When $t = 0, x = 0$: $c = -2\ln 20$

$t = 2\ln 20 - 2\ln(20 - x) \quad \Rightarrow \quad t = 2\ln \frac{20}{20 - x}$

Rearranging: $\frac{20}{20 - x} = e^{\frac{t}{2}} \quad \Rightarrow \quad \frac{20 - x}{20} = e^{-\frac{t}{2}}$

$1 - \frac{x}{20} = e^{-\frac{t}{2}} \quad \Rightarrow \quad x = 20\left(1 - e^{-\frac{1}{2}t}\right)$

Differentiating wrt t: $v = \frac{dx}{dt} = 10e^{-\frac{1}{2}t}$

Acceleration as a function of displacement

E.g. 4 Let $a = a(x)$. Given that $v\frac{dv}{dx} = a(x)$, find an expression for v^2 in terms of a.

Acceleration as a function of velocity

If acceleration is a function of velocity, v, then:

$$a(v) = \frac{dv}{dt}$$

By separating the variables find an expression for t.

$$\int 1 dt = \int \frac{1}{a(v)} dv$$
$$t = \int \frac{1}{a(v)} dv$$

Alternatively: $a(v) = v\frac{dv}{dx}$

By separating the variables find an expression for x.

$$\int 1 dx = \int \frac{v}{a(v)} dv$$
$$x = \int \frac{v}{a(v)} dv$$

The key is choosing which version of the differential equations to use.

E.g. 5 A cyclist and her bicycle have total mass 100 kg. She is working at a constant power of 80 watts. Calculate how far she travels in increasing her speed from 4 m/s to 8 m/s long a level road,

(a) if air resistance is neglected (give your answer exactly)

(b) making allowance for air resistance of $0.8v$ N when her speed is v m/s (give your answer to 3 s.f.).

Video (password needed): **Force as a function of time**

Video (password needed): **Force as a function of displacement**

Video (password needed): **Force as a function of velocity (Example 1)**

Video (password needed): **Force as a function of velocity (Example 2)**

Solutions to Starter and E.g.s

Exercise

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Summary

Important relationships: $v = \frac{dx}{dt}$ $\quad a = \frac{dv}{dt}$ $\quad a = v\frac{dv}{dx}$

Velocity as a function of displacement: $v(x) = \frac{dx}{dt} \Rightarrow t = \int \frac{1}{v(x)} dx$

Acceleration as a function of displacement: $v\frac{dv}{dx} = a(x) \Rightarrow \frac{1}{2}v^2 = \int a(x) dx$

Acceleration as a function of velocity: $a(v) = \frac{dv}{dt} \Rightarrow t = \int \frac{1}{a(v)} dv$

...or... $a(v) = v\frac{dv}{dx} \Rightarrow x = \int \frac{v}{a(v)} dv$