Distributions of related continuous random variables

Starter

1. **(Review of last lesson)** The continuous random variable T has probability density function $f(t)$, where $f(t) = \begin{cases} 5e^{-5t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find the probability that $T > E(T)$

(b) Find the value of the constant c such that $P(T > c) = 0.05$

Notes

When the distribution of a continuous random variable is given in terms of another continuous random variable, the cumulative distribution is calculated first and from there the probability density function can be found.

Here is an example to illustrate the method.

E.g. 1 The continuous random variable X has probability density function $f(x) = \begin{cases} 1.5\sqrt{x} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. The random variable Y is given by $Y = \dfrac{1}{\sqrt{X}}$.

Find the cumulative distribution function of Y and hence the probability distribution function.

Working:

$$F(x) = \int_0^x 1.5\sqrt{x}\,dx = \left[x^{\frac{3}{2}}\right]_0^x = x^{\frac{3}{2}}$$

So $F(x) = \begin{cases} 0 & x \leq 0 \\ x^{\frac{3}{2}} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$

Let $G(y)$ be the cumulative distribution function of Y.

$$G(y) = P(Y \leq y)$$

Since $Y = \dfrac{1}{\sqrt{X}}$***:***

$$G(y) = P\left(\frac{1}{\sqrt{X}} \leq y\right)$$

Rearrange to make X the subject of the inequality:

$$G(y) = P\left(X \geq \frac{1}{y^2}\right)$$

Put it in terms of $P(X \leq \ldots)$:

$$G(y) = 1 - P\left(X \leq \frac{1}{y^2}\right)$$

Replace $P(X \leq \ldots)$ by $F(\ldots)$:

$$G(y) = 1 - F\left(\frac{1}{y^2}\right)$$

Replace $F(\ldots)$ by its function:

From $F(x) = \begin{cases} 0 & x \leq 0 \\ x^{\frac{3}{2}} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$ we get

$$G(y) = 1 - F\left(\frac{1}{y^2}\right) = \begin{cases} 1 - 0 & \frac{1}{y^2} \leq 0 \\ 1 - \frac{1}{y^3} & 0 < \frac{1}{y^2} \leq 1 \\ 1 - 1 & \frac{1}{y^2} > 1 \end{cases}$$

$$G(y) = \begin{cases} 1 & \frac{1}{y^2} \leq 0 \\ 1 - \frac{1}{y^3} & y \geq 1 \\ 0 & y < 1 \end{cases}$$

i.e. $G(y) = \begin{cases} 1 - \frac{1}{y^3} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$

The pdf of Y:
$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{d}{dy}\left(1 - \frac{1}{y^3}\right) \\ &= \frac{d}{dy}\left(1 - y^{-3}\right) \\ &= 3y^{-4} \\ &= \frac{3}{y^4} \end{aligned}$$

The probability distribution function of Y is $g(y) = \begin{cases} \frac{3}{y^4} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$

Success criteria

Suppose that the continuous random variable Y is defined in terms of the continuous random variable such that $Y = h(X)$.

1. Find $F(x)$ if it is not given in the question.
2. Define $G(y)$ as the cumulative distribution function such that $G(y) = P(Y \leq y)$.
3. Replace Y by $h(X)$.
4. Rearrange the inequality $h(X) \leq y$ so that X is the subject.
5. Ensure the expression is in the form $P(X \leq \ldots)$.
6. Replace $P(X \leq \ldots)$ by $F(\ldots)$, where $F(x)$ is the cumulative distribution function of X.
7. Find the probability distribution function of Y by $g(y) = G'(y)$.

N.B. Remember to include the limits for $G(y)$ and $g(y)$.

E.g. 2 The continuous random variable X has cumulative distribution given by

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{x^4} & x \geq 1 \end{cases}$$

(a) Find the cumulative distribution function, $G(y)$, of the random variable Y, where $Y = \dfrac{1}{X^2}$.

(b) Hence find the probability distribution function, $g(y)$.

(c) Find $E(\sqrt[3]{Y})$.

Explanation (password needed): **The cdf & pdf of a related variable**

Solutions to Starter and E.g.s

Exercise

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Summary

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1. Find $F(x)$ if it is not given in the question.
2. Define $G(y)$ as the cumulative distribution function such that $G(y) = P(Y \leq y)$.
3. Replace Y by $h(X)$.
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7. Find the probability distribution function of Y by $g(y) = G'(y)$.

N.B. Remember to include the limits for $G(y)$ and $g(y)$.