Exact 1st order linear differential equations

Starter

1. (Review of last lesson) Calculate the exact value of the mean value of the function $y = \dfrac{4}{x^2+16}$ over the interval $0 \le x \le \pi$.

2. Solve these differential equations:

(a) $\dfrac{dy}{dx} = 2x + 5$ (b) $(x-3)\dfrac{dy}{dx} = y$

Notes

Terminology

Differential equations have an ***independent variable*** (the denominator of the derivative) and at least one ***dependent variable*** (the numerator of the derivative)

E.g. For $\dfrac{dy}{dx} = x^2 + y$ x is the independent variable and y is the dependent variable

The ***order of a differential equation*** is the largest number of times the dependent variable is differentiated.

E.g. The differential equation $\dfrac{d^4y}{dx^4} + 5\dfrac{dy}{dx} + 9y = 0$ has order 4.

Linear differential equations are where the ***dependent variable*** is raised to the power of 1 (including products of dependent variable with its derivatives)

E.g. $\dfrac{d^2y}{dx^2} + 9y = \sin x$ is linear but $y\dfrac{dy}{dx} + y = e^x$ is not because of the term $y\dfrac{dy}{dx}$.

General solution — solve the DE and express the answer in terms of the ***constant of integration***

Particular solution — use the given boundary conditions to find the constant of integration

Solving exact 1st order differential equations

From the starter, we can already solve differential equations of the form $\dfrac{dy}{dx} = f(x)$ (by direct integration) and $\dfrac{dy}{dx} = f(x)g(y)$ (by separation of variables). What if the differential equation cannot be reduced to either of these forms?

E.g. 1 Consider the left-hand side of these differential equations. Decide what their connection is with the product rule and hence solve the equations.

(a) $3x^2y + x^3\dfrac{dy}{dx} = x^4$ (b) $x\dfrac{dy}{dx} + y = e^x$

(c) $2ye^x\dfrac{dy}{dx} + e^xy^2 = e^{2x}$ (d) $x^2\cos y\dfrac{dy}{dx} + 2x\sin y = \dfrac{1}{x^2}$

Working: (a) The left-hand side is $\dfrac{d(x^3y)}{dx}$ so $\dfrac{d(x^3y)}{dx} = x^4$

$\Rightarrow \quad x^3y = \displaystyle\int x^4 dx \quad \Rightarrow \quad 5x^3y = x^5 + c$

Exact first order differential equations are of the form:

$$f(x)g'(y)\frac{dy}{dx}+f'(x)g(y)=Q(x) \quad \Rightarrow \quad \frac{d[f(x)g(y)]}{dx}=Q(x) \quad \Rightarrow \quad f(x)g(y)=\int Q(x)dx$$

E.g. 2 Solve these exact first order differential equations:

(a) $r^3\sec^2\theta+3r^2\tan\theta\dfrac{dr}{d\theta}=2\sin^2\theta$ (b) $\ln y+\dfrac{x}{y}\dfrac{dy}{dx}=\sec x\tan x$

Working: (a)

$$r^3\sec^2\theta+3r^2\tan\theta\frac{dr}{d\theta}=2\sin^2\theta$$

$$\frac{d(r^3\tan\theta)}{d\theta}=2\sin^2\theta$$

$$r^3\tan\theta=\int 2\sin^2\theta d\theta=\int(1-\cos 2\theta)d\theta$$

$$r^3\tan\theta=\theta-\frac{1}{2}\sin 2\theta+c$$

(b)

$$\ln y+\frac{x}{y}\frac{dy}{dx}=\sec x\tan x$$

$$\frac{d(x\ln y)}{dx}=\sec x\tan x$$

$$x\ln y=\int\sec x\tan x dx$$

$$x\ln y=\sec x+c$$

Video: Exact first order linear differential equations

Solutions to Starter and E.g.s

Exercise
FP2/3 p224 Ex 1A Qu 1, 2

Summary
Exact first order differential equations are of the form:

$$f(x)g'(y)\frac{dy}{dx}+f'(x)g(y)=Q(x) \quad \Rightarrow \quad \frac{d[f(x)g(y)]}{dx}=Q(x) \quad \Rightarrow \quad f(x)g(y)=\int Q(x)dx$$