Exponential distribution

Starter

1. (Review of last lesson) A random variable X has distribution $X \sim \text{R}(-1, 1)$.
   (a) Write down the probability density function, $f(x)$.
   (b) Find the cumulative distribution function, $F(x)$.

2. The random variable X has probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.
   (a) Sketch the graph of $f(x)$.
   (b) Show that $f(x)$ satisfies the requirements for a probability density function.
   (c) Find expressions for the mean and the variance in terms of λ.

Notes

A continuous random variable of the form $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ is called an ***exponential distribution*** and can be denoted $X \sim \text{Exp}(\lambda)$.

The mean and variance are given by $\text{E}(X) = \dfrac{1}{\lambda}$ and $\text{Var}(X) = \dfrac{1}{\lambda^2}$ respectively — these appear in the formula booklet.

E.g. 1 The continuous random variable X is such that $X \sim \text{Exp}(\lambda)$. Find expressions in terms of λ for:
(a) $P(X < x)$
(b) the cumulative distribution function, $F(x)$.
(c) $P(x_1 \leq X \leq x_2)$

Working:

(a) $$P(X < x) = \int_0^x \lambda e^{-\lambda t} dt = \left[e^{-\lambda t}\right]_x^0 = 1 - e^{-\lambda x}$$

(b) $$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

(c) $$\begin{aligned} P(x_1 \leq X \leq x_2) &= P(X \leq x_2) - P(X \leq x_1) \\ &= \left(1 - e^{-\lambda x_2}\right) - \left(1 - e^{-\lambda x_1}\right) \\ &= e^{-\lambda x_1} - e^{-\lambda x_2} \end{aligned}$$

The cumulative distribution function for $X \sim \text{Exp}(\lambda)$ is $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$.

E.g. 2 A continuous random variable X has $X \sim \text{Exp}(0.2)$. Find:
(a) $\text{E}(X)$ and $\text{Var}(X)$
(b) State the cumulative distribution and hence find $P(3.5 \leq X \leq 4.5)$ to

E.g. 3 A random variable is modelled using an exponential distribution with mean equal to 1.
(a) Obtain the cumulative distribution function, $F(x)$.
(b) By appropriate use of $F(x)$, determine the exact value of:
   (i) the median
   (ii) the probability that a random observation is between the median and the mean.

E.g. 4 The manufacturer of bicycle tyres believe that the distance between punctures can be modelled by the random variable, X km, with p.d.f. $f(x) = \begin{cases} 0.005e^{-0.005x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Finds the mean distance between punctures.

(b) A cyclist has just repaired a puncture. Calculate the probability she will be able to ride at least 500 km before having another puncture.

(c) Another cyclist has just repaired a puncture. Calculate the probability he will be able to ride less than 30 km before having another puncture.

(d) A third cyclist starts a race with new tyres but then has a puncture after 30 km. When she starts again, she has another puncture after k km. She contacts the manufacturer who state that the probability of the combined probability of the punctures is 0.005. What is the value of k?

Video: Exponential distribution
Video: Exponential distribution

Solutions to Starter and E.g.s

Exercise
p141 7G Qu 1i, 2i, 3-7, (8, 9, 11 red)

Summary
For a continuous random variable X that is distributed exponentially $X \sim \text{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{E}(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

$$P(X < x) = 1 - e^{-\lambda x}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$