Further factorising

Starter

1. (Review of last lesson) Let 1, ω and ω^2 be the cube roots of unity. Find the equation whose roots are $\frac{1}{3}$, $\frac{1}{2+\omega}$ and $\frac{1}{2+\omega^2}$.

2. (a) Simplify $(p+\mathrm{i}q)+(p-\mathrm{i}q)$
   (b) Expand $(p+\mathrm{i}q)(p-\mathrm{i}q)$.
   (b) Find the product $\big(x-(p+\mathrm{i}q)\big)\big(x-(p-\mathrm{i}q)\big)$.

3. Find the product of $\big(x-r(\cos\theta+\mathrm{i}\sin\theta)\big)\big(x-r(\cos\theta-\mathrm{i}\sin\theta)\big)$.

Notes

We know that complex roots come in conjugate pairs i.e. if $p+\mathrm{i}q$ is a root then so is $p-\mathrm{i}q$. Therefore, the factors of a polynomial are either linear $(x-a)$ or complex conjugate pairs $\big(x-(p+\mathrm{i}q)\big)\big(x-(p-\mathrm{i}q)\big)$.

From the starter, each ***complex conjugate pair of factors forms a real quadratic factor***.

So ***every polynomial can be expressed as the product of linear and quadratic factors with real coefficients***.

Therefore, every polynomial with ***odd degree*** and real coefficients must have ***at least one linear factor***.

Roots are $p \pm \mathrm{i}q$ $\Rightarrow$ quadratic factor is $x^2-2px+p^2+q^2$
Roots are $r(\cos\theta \pm \mathrm{i}\sin\theta)$: $\Rightarrow$ quadratic factor is $x^2-(2r\cos\theta)x+r^2$
Roots are $re^{\pm\mathrm{i}\theta}$ $\Rightarrow$ quadratic factor is $x^2-(2r\cos\theta)x+r^2$

E.g. 1 (a) Solve the equation $x^6 - 2x^3 + 4 = 0$, expressing the roots in the form $r(\cos\theta \pm \mathrm{i}\sin\theta)$.

(b) Plot the roots on an Argand diagram and describe their positions.

(c) Hence express $x^6 - 2x^3 + 4$ as the product of three quadratic factors with real coefficients.

Working: (a) ***By the quadratic formula:*** $x^3 = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 4}}{2 \times 1}$

$$x^3 = 1 \pm \mathrm{i}\sqrt{3} = 2e^{\pm \mathrm{i}\frac{\pi}{3}}$$

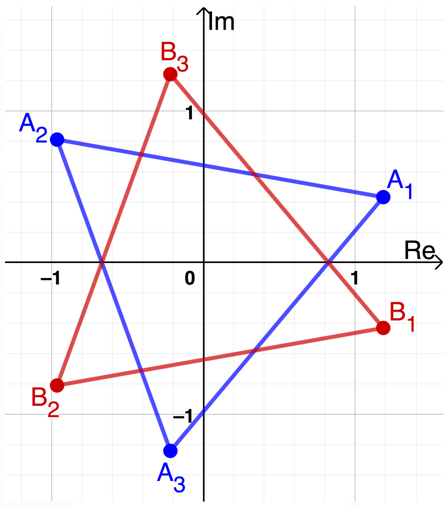
Find $\dfrac{2\pi}{n}$: $\dfrac{2\pi}{3}$

From $x^3 = 1 + \mathrm{i}\sqrt{3} = 2e^{\mathrm{i}\frac{\pi}{3}}$: $x = 2^{\frac{1}{3}}e^{\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}e^{\mathrm{i}\frac{7\pi}{9}}$ and $2^{\frac{1}{3}}e^{-\mathrm{i}\frac{5\pi}{9}}$

From $x^3 = 1 - \mathrm{i}\sqrt{3} = 2e^{-\mathrm{i}\frac{\pi}{3}}$: $x = 2^{\frac{1}{3}}e^{-\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}e^{-\mathrm{i}\frac{7\pi}{9}}$ and $2^{\frac{1}{3}}e^{\mathrm{i}\frac{5\pi}{9}}$

The roots are $2^{\frac{1}{3}}\left(\cos\dfrac{\pi}{9} \pm \mathrm{i}\sin\dfrac{\pi}{9}\right)$, $2^{\frac{1}{3}}\left(\cos\dfrac{5\pi}{9} \pm \mathrm{i}\sin\dfrac{5\pi}{9}\right)$ and $2^{\frac{1}{3}}\left(\cos\dfrac{7\pi}{9} \pm \mathrm{i}\sin\dfrac{7\pi}{9}\right)$.

(b) The solutions of the two separate complex numbers are equally spaced around two circles on separate diagrams but the six solutions are not equally spaced around a single circle.



(c) Each pair of conjugate roots form a quadratic factor.

$$\left(x - 2^{\frac{1}{3}}e^{\mathrm{i}\frac{\pi}{9}}\right)\left(x - 2^{\frac{1}{3}}e^{-\mathrm{i}\frac{\pi}{9}}\right) = x^2 - \left(2^{\frac{4}{3}}\cos\frac{\pi}{9}\right)x + 2^{\frac{2}{3}}$$

$$\left(x - 2^{\frac{1}{3}}e^{\mathrm{i}\frac{5\pi}{9}}\right)\left(x - 2^{\frac{1}{3}}e^{-\mathrm{i}\frac{5\pi}{9}}\right) = x^2 - \left(2^{\frac{4}{3}}\cos\frac{5\pi}{9}\right)x + 2^{\frac{2}{3}}$$

$$\left(x - 2^{\frac{1}{3}}e^{\mathrm{i}\frac{7\pi}{9}}\right)\left(x - 2^{\frac{1}{3}}e^{-\mathrm{i}\frac{7\pi}{9}}\right) = x^2 - \left(2^{\frac{4}{3}}\cos\frac{7\pi}{9}\right)x + 2^{\frac{2}{3}}$$

$$\therefore x^6 - 2x^3 + 4 = \left(x^2 - \left(2^{\frac{4}{3}}\cos\frac{\pi}{9}\right)x + 2^{\frac{2}{3}}\right)\left(x^2 - \left(2^{\frac{4}{3}}\cos\frac{5\pi}{9}\right)x + 2^{\frac{2}{3}}\right)\left(x^2 - \left(2^{\frac{4}{3}}\cos\frac{7\pi}{9}\right)x + 2^{\frac{2}{3}}\right)$$

E.g. 2 Express $x^6 - 4x^3 + 8 = 0$ as the product of three quadratic factors, where coefficients are given in terms of integers or roots.

E.g. 3 (a) Express $x^5 - 1$ in the form $(x-1)(ax^4 + bx^3 + cx^2 + dx + e)$ where a, b, c, d and e are integers to be found.

(b) Hence, find the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$ and express $x^4 + x^3 + x^2 + x + 1$ as the product of quadratic factors.

N.B. $x^n - a^n \equiv (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-2} + a^3xn - 2 + \ldots + a^{n-2}x + a^{n-1})$
When $a = 1$ we get $x^n - 1 \equiv (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$

E.g. 4 Solve the equation $(x-2)^3 - 1 = 0$ and hence express $(x-2)^3 - 1 = 0$ as the product of a linear and a quadratic factor.
Hint: solve for $x - 2$.

No Video:

Solutions to Starter and E.g.s

Exercise
p45 2E Qu 1-6

Summary
Every polynomial can be expressed as the ***product of linear and quadratic factors*** with ***real coefficients***.

Roots are $p \pm \mathrm{i}q$ $\Rightarrow$ quadratic factor is $x^2 - 2px + p^2 + q^2$
Roots are $r(\cos\theta \pm \mathrm{i}\sin\theta)$: $\Rightarrow$ quadratic factor is $x^2 - (2r\cos\theta)x + r^2$
Roots are $re^{\pm\mathrm{i}\theta}$ $\Rightarrow$ quadratic factor is $x^2 - (2r\cos\theta)x + r^2$

$x^n - a^n \equiv (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-2} + a^3xn - 2 + \ldots + a^{n-2}x + a^{n-1})$
When $a = 1$ we get $x^n - 1 \equiv (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$