Homogenous second order linear differential equations

Starter

1. (Review of last lesson)

   Solve the equation $x\dfrac{dy}{dx} + 2y = \dfrac{\cos x}{x}$ given that $y = 3$ when $x = \pi$.

2. Solve the equation $a\dfrac{dy}{dx} + by = 0$. Write down what you notice about the coefficient of x in the solution.

Notes

Homogenous differential equations are when the sum of the terms involving the dependent variable is zero i.e. the right-hand side is zero.

Homogenous first order linear differential equations
Consider the homogeneous first order linear differential equation with constant coefficients:

$$a\frac{dy}{dx} + by = 0$$

From the starter its solution is $y = Ae^{\lambda x}$ where λ is the root of the equation $a\lambda + b = 0$ and A is an arbitrary constant.

N.B. $a\lambda + b = 0$ is called the ***auxiliary equation***.

Homogenous second order linear differential equations
Consider the homogeneous second order linear differential equation with constant coefficients:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

When is $y = Ae^{\lambda x}$ a solution to the equation above?

E.g. 1 Assume $y = Ce^{\lambda x}$ is the solution for the second order differential equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$. Find the value(s) of λ for which this works.

Working: $y = Ce^{\lambda x} \quad \Rightarrow \quad \dfrac{dy}{dx} = C\lambda e^{\lambda x} \quad \Rightarrow \quad \dfrac{d^2y}{dx^2} = C\lambda^2 e^{\lambda x}$

$\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$: $\quad C\lambda^2 e^{\lambda x} - 5C\lambda e^{\lambda x} + 6Ce^{\lambda x} = 0$

Divide by $Ce^{\lambda x}$ $\quad \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 2$ or $\lambda = 3$

So $y = Ce^{\lambda x}$ is a solution when $\lambda = 2$ and $\lambda = 3$ i.e. $y = Ce^{2x}$ and $y = Ce^{3x}$

Since both $y = Ce^{2x}$ and $y = Ce^{3x}$ each satisfy the equation $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ then $y = Ae^{2x} + Be^{3x}$ is also a solution of the differential equation.

Non-homogenous second order differential equations

The ***complementary function*** of the non-homogenous differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=q(x)$ is the function that satisfies $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

Auxiliary equation for second order linear differential equations with constant coefficients

The ***auxiliary equation*** of the second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=q(x)$ is $a\lambda^2+b\lambda+c=0$.

When $b^2-4ac>0$, there are two distinct roots, say λ_1 and λ_2.

The ***complementary function*** of the differential equation is $y=Ae^{\lambda_1 x}+Be^{\lambda_1 x}$.

E.g. 2 Find the complementary function for the differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=2x$

Proof when $b^2-4ac>0$

Let α and β be the roots of the equation $a\lambda^2+b\lambda+c=0$ so $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Then $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ can be rewritten as $a\frac{d^2y}{dx^2}-(\alpha+\beta)\frac{dy}{dx}+\alpha\beta y=0$

Expanding and rearranging:
$$\frac{d^2y}{dx^2}-\beta\frac{dy}{dx}=\alpha\left(\frac{dy}{dx}-\beta y\right)$$
$$\frac{d}{dx}\left(\frac{dy}{dx}-\beta y\right)=\alpha\left(\frac{dy}{dx}-\beta y\right)$$

Let $u=\frac{dy}{dx}-\beta y$:
$$\frac{du}{dx}=\alpha u.$$
$$\int\frac{1}{u}du=\int\alpha dx$$
$$\ln|u|=\alpha x+c$$
$$u=Ce^{\alpha x}$$

Substitute into $u=\frac{dy}{dx}-\beta y$:
$$\frac{dy}{dx}-\beta y=Ce^{\alpha x}\quad(1)$$

We need to eliminate $\frac{dy}{dx}$ to get an expression for y.

Swapping α and β around:
$$\frac{dy}{dx}-\alpha y=De^{\beta x}\quad(2)$$

Eliminating $\frac{dy}{dx}$ from (1) and (2):
$$(\alpha-\beta)y=Ce^{\alpha x}-De^{\beta x}.$$
$$y=\frac{C}{\alpha-\beta}e^{\alpha x}-\frac{D}{\alpha-\beta}e^{\beta x}$$
$$y=Ae^{\alpha x}+Be^{\beta x}$$

What is the next case to look at?

Proof when $b^2 - 4ac = 0$
Assume the auxiliary equation, $a\lambda^2 + b\lambda + c = 0$, has equal roots, α.
Using the same working as above: $\frac{dy}{dx} - \alpha y = Ce^{\alpha x}$

There is no second equation to eliminate $\frac{dy}{dx}$ because we cannot simply swap α and β around.
Hence, we need to solve the differential equation $\frac{dy}{dx} - \alpha y = Ce^{\alpha x}$.

E.g. 3 Solve the equation $\frac{dy}{dx} - \alpha y = Ce^{\alpha x}$ and hence give the complementary function of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$ when $b^2 - 4ac = 0$.

Working: *Integrating factor:* $I(x) = e^{-\int \alpha dx} = e^{-\alpha x}$

Multiply by the integrating factor: $e^{-\alpha x}\frac{dy}{dx} - \alpha e^{-\alpha x}y = C$

$$\frac{d(e^{-\alpha x}y)}{dx} = C$$

$$e^{-\alpha x}y = \int C dx$$

$$e^{-\alpha x}y = Cx + D$$

So the complementary function is $y = (Cx + D)e^{\alpha x}$

For second order linear DE of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$ the ***auxiliary equation*** is $a\lambda^2 + b\lambda + c = 0$.

Discriminant	Root(s)	Complementary function
$b^2 - 4ac > 0$	α and β	$y = Ae^{\alpha x} + Be^{\beta x}$
$b^2 - 4ac = 0$	α (repeated)	$y = (Cx + D)e^{\alpha x}$

E.g. 4 Find the complementary function for $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$.

Working: *Auxiliary equation:* $\lambda^2 - 4\lambda + 4 = 0$

Solving: $(\lambda - 2)^2 = 0$

$\lambda = 2$ (repeated)

Since roots are repeated, the complementary function is $y = (Cx + D)e^{2x}$.

E.g. 5 Find the complementary function of the second order differential equations:

(a) $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = \cos 4x$ (b) $3\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 4y = 5x^3$

Video: Homogenous linear second order differential equations

Solutions to Starter and E.g.s

Exercise
p234 10C Qu 1, 3, 4, 5, 7, 8, 11

Summary

For a second order linear differential equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$ the ***auxiliary equation*** is $a\lambda^2 + b\lambda + c = 0$.

The ***complementary function*** is a solution to the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

Discriminant	Root(s)	Complementary function
$b^2 - 4ac > 0$	α and β	$y = Ae^{\alpha x} + Be^{\beta x}$
$b^2 - 4ac = 0$	α (repeated)	$y = (Cx + D)e^{\alpha x}$