Hypothesis testing for the mean of a large sample

Starter

1. **(Review of last lesson)** A large number of samples of size n is taken from $X \sim \text{Po}(8)$ and approximately $6.2\,\%$ of the sample means are less than 7.5. Estimate the value of n.

2. **(Review of previous material)** Experience has shown that the scores obtained in a particular test are normally distributed with mean score 70 and variance 36. When the test is taken by a random sample of 49 students, the mean score is 68.5. Is there sufficient evidence, at the $4\,\%$ level, that these students have not performed as well as expected?

Notes

In A2 maths, we carried a hypothesis test on a sample taken from a population that was normally distributed to test whether the sample could come from that population. To do this we needed to be given the variance of the population.

Using the central limit theorem, hypothesis tests can be carried out any type of distribution so long as $n > 25$ i.e. it does not need to be given that the distribution is normal.

E.g. 1 A farmer sells eggs claiming their masses have a mean average of 60 g. The supermarket believes the eggs are underweight and collects a random sample of 85 eggs. The sample mean is 58.4 g. Given that the eggs are known to have a variance of 49 g², test the farmer's claim at the $5\,\%$ significance level. State whether you used the central limit theorem and explain your answer.

Working: $\mu = 60, \sigma^2 = 49$

Since the distribution is not stated but the sample is large enough, the central limit theorem is used to get: $\bar{X}_{85} \sim \text{N}\left(60, \dfrac{49}{85}\right)$

$H_0 : \mu = 60$ g (the mass of the eggs is as the farmer states)
$H_1 : \mu < 60$ (the eggs are underweight)
$n = 85, \bar{x} = 58.4$, level of significance is $5\,\%$

$p-$value method

$P(\bar{X} < 58.4) = 0.0175 \equiv 1.75\,\%$

Since $1.75\,\% < 5\,\%$, $\bar{x} = 58.4$ ***lies*** in the critical region.
Therefore, the supermarket ***rejects*** H_0 and concludes that there is evidence to suggest the eggs are underweight.

Critical value method

Let x_{cv} be the critical value such that $P(\bar{X} < x_{cv}) = 0.05 \Rightarrow$
$x_{cv} = 58.75$
Since $\bar{x} = 58.4 < 58.75 = x_{cv}$, we reject H_0 and conclude that there is evidence to suggest the eggs are underweight.

E.g. 2 A cyclist regularly cycles a route with a mean speed of 25.3 km/h and a standard deviation of 3 km/h. After a period of specific strength training, she rides the same route 30 times and averages 26.1 km/h.

(a) Using the information given, test at the $10\,\%$ significance level whether the strength training has improved the speed of the cyclist.
(b) State whether you used the central limit theorem and explain your answer.
(c) Make one criticism of the hypothesis test and suggest an improvement based on your criticism.

Population variance is unknown

Furthermore, if the variance of the population is unknown, it can be estimated from the sample using:

$$s^2 = \frac{n}{n-1} \times \text{sample variance} \qquad \Rightarrow \qquad s^2 = \frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$

E.g. 3 At a supermarket, mince meat is sold in 1.2 kg packs. Trading standards inspectors take a random sample of 14 packs and the masses were measured. They obtained the results that $\sum x_i = 15.764$ and $\sum x_i^2 = 17.9715$.

(a) State an assumption that must be made before a hypothesis test can be carried out.

(b) Test at the $2\,\%$ level whether the packs are significantly underweight.

(c) If the case went to court, what mathematical defence could the supermarket's lawyer use?

Working: (a) The assumption is that the mass of the packs of mince meat follow a normal distribution since the sample is too small to use the central limit theorem.

(b) $\bar{x} = \dfrac{15.764}{14} = 1.126$ kg

Since σ^2 is unknown, it must be calculated from the sample.

$$s^2 = \frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$
$$= \frac{14}{13}\left(\frac{17.9715}{14} - \left(\frac{15.764}{14}\right)^2\right)$$
$$= 0.01702$$

$X \sim \text{N}(1.2, \sigma^2) \qquad \Rightarrow \qquad \bar{X}_{14} \sim \text{N}\left(1.2, \dfrac{0.01702}{14}\right)$

$H_0 : \mu = 1.2$ (the mass of mince meat packs are correctly labelled)

$H_1 : \mu < 1.2$ (the mince meat packs are underweight)

$n = 14, \bar{x} = 1.126$, level of significance is $2\,\%$

$p-$value method

$P(\bar{X} < 1.126) = 0.0169 \equiv 1.69\,\%$

Since $1.69\,\% < 2\,\%$, $\bar{x} = 1.126$ ***does lie*** in the critical region.

Therefore, we ***reject*** H_0 and conclude that there is evidence to suggest the packs of mince meat are underweight.

Critical value method

Let x_{cv} be the critical value such that $P(\bar{X} < x_{cv}) = 0.02 \Rightarrow$ $x_{cv} = 1.1284$

Since $\bar{x} = 1.1256 < 1.1284 = x_{cv}$, we reject H_0 and conclude that there is evidence to suggest the packs of mince meat are underweight.

(c) The sample is small so that the sample variance will not be close to the variance of the population.

E.g. 4 A machine is set to produce metal rods with length 4.1 cm. After the machine is serviced, the engineer wants to check whether the machine is working correctly and takes a random sample of 30 rods and finds that $\sum x_i = 128.441$ and $\sum x_i^2 = 554.816$. Test at the $5\,\%$ level whether the machine is working correctly using:

(a) the previous variance of the machine, which was 0.27 cm

(b) a new value for the variance based on the sample.

Video: Introduction to hypothesis testing using the Central Limit Theorem

Solutions to Starter and E.g.s

Exercise

p169 9A Qu 1i, 2i, 3i, 4-8, (9 red)

Summary

Using the Central Limit Theorem, hypothesis tests can be carried out any type of distribution so long as $n > 25$.

If the variance of the population is unknown, it can be estimated from the sample using:

$$s^2 = \frac{n}{n-1} \times \text{sample variance} \quad \Rightarrow \quad s^2 = \frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$

Using a calculator with the Normal distribution:

Probabilities Menu >> 7:Distribution >> 2:Normal CD

Input screen **Normal CD**

- **Lower:** type a large number like -99999 if $P(X < x)$
- **Upper:** type a large number like 99999 if $P(X > x)$
- σ: square root to get standard deviation if $X \sim \text{N}(5, 7)$
- μ: *you may need to scroll down to see this*

$x-$***values*** Menu >> 7:Distribution >> 3:Inverse Normal

Input screen **Inverse normal**

- **Area:** type a large number between 0 and 1
- σ: square root to get standard deviation if $X \sim \text{N}(5, 7)$
- μ: