Induction and series

Starter

1. **(Review of previous material)**

   Write down the series of numbers and their total as denoted by $\sum_{r=1}^{4}(5r+2)$.

2. **(Review of previous material)** Find the value of $\sum_{n=1}^{\infty} 7 \times \left(\frac{1}{3}\right)^n$

3. Simplify $\frac{1}{4}k^2(k+1)^2 + (k+1)^3$ into one algebraic expression.

Notes

The method of proving by induction is the same with series but the key is to add the next term in the sequence to the given formula for the sequence. Then factorise the formula in such a way to show that the formula is same but with k replaced by $k+1$.

Here are four stages to proof by induction:

1. ***(Proposition)*** Statement of the proposition $P(n)$
2. ***(Prove the basic case)*** Prove the result for $n = 1$.
3. ***(Inductive steps)*** Assume the result is true for k and then prove that if the result for k is true then the result is also true for $k+1$
4. ***(Completion of the proof)*** "But this is $P(k)$ with k replaced by $k+1$. Therefore, if $P(k)$ is true, then $P(k+1)$ is also true. $P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers."

E.g. 1 Prove by induction that $\sum_{r=1}^{n}(3r-1)=\frac{1}{2}n(3n+1)$.

Working: ***(Proposition)***

Let $P(n)$ be the proposition that $\sum_{r=1}^{n}(3r-1)=\frac{1}{2}n(3n+1)$.

(Prove the basic case)

When $n=1$, LHS $=3\times1-1=2$ and RHS $=\frac{1}{2}\times1\times(3\times1+1)=2$

Therefore $P(1)$ is true.

(Inductive step — n replaced by k)

Assume that $P(k)$ is true i.e. $\sum_{r=1}^{k}(3r-1)=\frac{1}{2}k(3k+1)$

(Inductive step — consider the next term)

Add the next term to both sides:

$$P(k+1)=\sum_{r=1}^{k+1}(3r-1)$$

$$=\frac{1}{2}k(3k+1)+3(k+1)-1$$

(Inductive step — manipulation to show the formula is the same)

$$=\frac{1}{2}\left(3k^2+k+6k+6-2\right)$$

$$=\frac{1}{2}(3k^2+7k+4)$$

$$=\frac{1}{2}(k+1)(3k+4)$$

$$=\frac{1}{2}(k+1)(3(k+1)+1)$$

(Completion)

But this is $P(k)$ with k replaced by $k+1$.
Therefore, if $P(k)$ is true, then $P(k+1)$ is also true.
$P(1)$ is true and if $P(k)$ is true, then $P(k+1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers.

E.g. 2 Prove by induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$.

E.g. 3 Prove by induction that $1\times2+2\times3+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

Video: Proof by induction (sum of squares)
Video: Proof by induction (sum of cubes)
Video: Proof by induction (other series)

Solutions to Starter and E.g.s

Exercise
p9 1B Qu 1-7

Summary
Here are four stages to proof by induction:

1. ***(Proposition)*** Statement of the proposition $P(n)$
2. ***(Prove the basic case)*** Prove the result for $n = 1$.
3. ***(Inductive steps)*** Assume the result is true for k and then prove that if the result for k is true then the result is also true for $k + 1$
4. ***(Completion of the proof)*** "But this is $P(k)$ with k replaced by $k + 1$. Therefore, if $P(k)$ is true, then $P(k + 1)$ is also true. $P(1)$ is true and if $P(k)$ is true, then $P(k + 1)$ is true. Using the principle of mathematical induction, $P(n)$ is true for all positive integers."