Non-homogenous second order linear differential equations

Starter

1. (Review of last lesson)

Find the complementary function for these differential equations:

(a) $25\frac{d^2y}{dx^2}+20\frac{dy}{dx}+4=\sinh x$ (b) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=3x-5$

2. Find the general solution of $\cos x\frac{dy}{dx}+y\sin x=\tan x$.

Notes

Non-homogenous differential equations vs. homogenous differential equations

A ***non-homogenous differential equation*** is when a e.g. $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=q(x)$

Homogenous differential equations are when the sum of the terms involving the dependent variable is zero i.e. the right-hand side is zero. E.g. $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

Complementary functions and particular integrals

The solution of the differential equation $\cos x\frac{dy}{dx}+y\sin x=\tan x$ has two parts: $\frac{1}{2}\sec x$ and $C\cos x$.

Let $y=\frac{1}{2}\sec x$ then

$$\cos x\frac{dy}{dx}+y\sin x=\cos x\times\frac{1}{2}\sec x\tan x+\frac{1}{2}\sec x\sin x=\frac{1}{2}\tan x+\frac{1}{2}\tan x=\tan x$$

So $y=\frac{1}{2}\sec x$ is a solution of the full differential equation $\cos x\frac{dy}{dx}+y\sin x=\tan x$.

It is called the ***particular integral***.

N.B. Sometimes the ***particular integral*** is called the ***trial function***.

Let $y=C\cos x$ then

$$\cos x\frac{dy}{dx}+y\sin x=\cos x\times(-C\sin x)+C\cos x\times\sin x=0$$

So $y=C\cos x$ is a solution of the equation $\cos x\frac{dy}{dx}+y\sin x=\tan x$.

It is called the ***complementary function***.

General solution

The general solution of a linear differential equation is the sum of a particular integral and a complementary function.

General solution = particular integral + complementary function

where the ***particular integral*** satisfies the given differential equation and,

the ***complementary function*** is the general solution of the equation with 0 in place of the terms which are independent of y.

When finding the general solution of a differential equation, firstly ***calculate the complementary function*** and ***then*** find ***the particular integral***..

Finding the particular integral

The ***particular integral***, sometimes called the ***trial function***, is found by looking at the function which does not involve the dependent variable and choosing a function that might work.

In general, if $q(x)$ is:

a polynomial, choose a polynomial of the same degree
a special function, choose a similar special function

E.g. 1 Solve the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

Working: ***Auxiliary equation:*** $\lambda^2 + 6\lambda + 5 = 0$

$(\lambda + 5)(\lambda + 1) = 0$

$\lambda = -1$ or $\lambda = -5$

The complementary function is $y = Ae^{-x} + Be^{-5x}$

Particular integral: $q(x)$ is a polynomial of degree 1 so try $y = ax + b$.

$\Rightarrow$ $y' = a$ and $y'' = 0$

N.B. Avoid using $y = Ax + B$ since the letters A and B are used in the complementary function.

Substitute: $0 + 6a + 5(ax + b) = 2x$

Equating coefficients:

x: $2 = 5a \Rightarrow a = 0.4$

constant: $6a + 5b = 0 \Rightarrow b = -0.48$

General solution is $y = Ae^{-x} + Be^{-5x} + 0.4x - 0.48$

When $x = 0$, $y = 0$: $0.48 = A + B$

When $x = 0$, $\frac{dy}{dx} = 0$: $0.4 = A + 5B$

Solving simultaneously: $A = 0.5, B = -0.02$

The solution is $y = 0.5e^{-x} - 0.02e^{-5x} + 0.4x - 0.48$

N.B. Notice that the general solution was found before substituting the boundary conditions to find A and B.

E.g. 2 Consider the differential equation $\frac{dy}{dx} + 4y = \sin 2x$.

(a) State the auxiliary equation and hence the complementary function.
(b) Show that the particular integral $y = a\sin 2x$ does not work.
(c) Show that the particular integral $y = a\cos 2x$ does not work.
(d) Find the particular integral in the form $y = a\sin 2x + b\cos 2x$ does not work. Hence state the general solution of the differential equation.

E.g. 3 Consider the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$.

(a) Find the complementary function.
(b) The normal procedure to find the particular integral is to try $y = ae^{3x}$. Show that $y = ae^{3x}$ does not work and suggest a reason why (you may wish to refer to the complementary function).
(c) Without calculation, but with reference to the complementary function, explain why $y = axe^{3x}$ would also not work.
(d) Suggest an alternative trial function and hence find the particular integral.

Function, $q(x)$	Particular integral
Polynomial, degree 0 or 1	Try $y = ax + b$
Polynomial, degree 2	Try $y = ax^2 + bx + c$
$k \sin nx$ or $k \cos nx$	Try $y = a \sin nx + b \cos nx$
ke^{nx}	Try $y = ae^{nx}$ or $y = axe^{nx}$ or $y = ax^2e^{nx}$

N.B. Check that the form of the particular integral is not already in the complementary function. If $y = F(x)$ does not work, try $y = xF(x)$.

E.g. 4 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4y = 3e^{-2x}$.

Video: Particular integrals where f(x) = k
Video: Particular integrals where f(x) is linear
Video: Particular integrals where f(x) is quadratic
Video: Particular integrals where f(x) is exponential
Video: Particular integrals where f(x) is trigonometric
Video: Special types of particular integrals
Video: Particular solutions involving boundary conditions

Exam questions: General solutions where the particular integral is linear
Exam questions: General solutions where the particular integral is exponential
Exam questions: General solutions where the particular integral is trigonometric
Exam questions: Particular solutions using boundary conditions

Solutions to Starter and E.g.s

Exercise
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Summary
General solution = particular integral + complementary function
where the ***particular integral*** satisfies the given differential equation and,
the ***complementary function*** is the general solution of the equation with 0 in place of the terms which are independent of y.
When finding the general solution of a differential equation, calculate the particular integral first and then find the particular integral.

Function, $q(x)$	Particular integral
Polynomial, degree 0 or 1	Try $y = ax + b$
Polynomial, degree 2	Try $y = ax^2 + bx + c$
$k \sin nx$ or $k \cos nx$	Try $y = a \sin nx + b \cos nx$
ke^{nx}	Try $y = ae^{nx}$ or $y = axe^{nx}$ or $y = ax^2e^{nx}$