Piecewise-defined probability density function

Starter

1. (Review of last lesson) Let X be a crv with pdf $f(x) = \begin{cases} \frac{1}{3}x & 0 \leq x \leq k \\ 0 & \text{otherwise} \end{cases}$.

   (a) Calculate the value of k.
   (b) Find the cumulative distribution function $F(x)$.

2. If X is a continuous random variable with pdf $f(x) = \begin{cases} \frac{1}{3}x & 0 \leq x \leq 2 \\ 2 - \frac{2}{3}x & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$.

   (a) Sketch the graph of $y = f(x)$.
   (b) Find the cumulative distribution function $F(x)$.
   (b) Sketch $y = F(x)$.

Notes

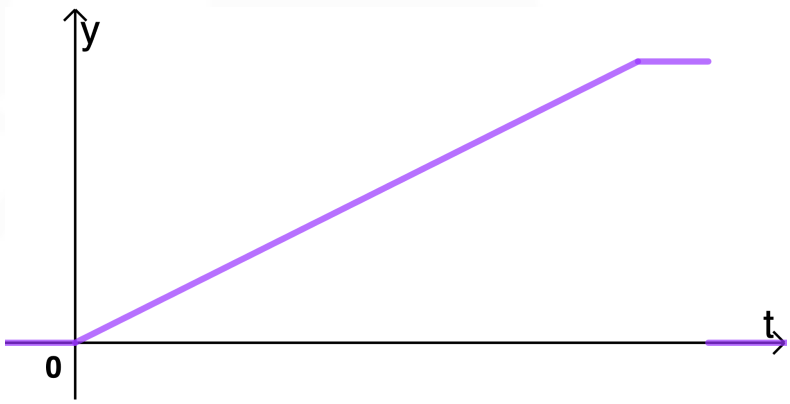
Piecewise functions have different functions defined for different parts of the domain. When a pdf is defined piecewise, the resulting cdf must still be a continuous function. The usual rules still apply but area calculations require ***more than one definite integral*** to be considered.

E.g. 1 A continuous random variable, T, has probability density function:

$$f(t) = \begin{cases} at & 0 \leq t \leq 8 \\ 8a & 8 < t \leq 9 \\ 0 & \text{otherwise} \end{cases}, \text{ where } a \text{ is a constant.}$$

(a) Sketch $f(t)$.
(b) Determine the value of the constant a.
(c) Obtain the cumulative distribution function, $F(t)$.
(d) Calculate the probability $P(T > 7)$.

Working: (a)



(b) $$\int_0^8 at\,dt + \int_8^9 8a\,dt = 1 \quad \Rightarrow \quad \left[\frac{1}{2}at^2\right]_0^8 + \left[8at\right]_8^9 = 1$$

$$\therefore 32a + 72a - 64a = 1 \quad \Rightarrow \quad a = \frac{1}{40} = 0.025$$

(c) $0 \le t \le 8$: $F(t) = \int_0^t \frac{1}{40} t dt = \left[\frac{1}{80}t^2\right]_0^t = \frac{1}{80}t^2$

$F(8) = \frac{1}{80} \times 8^2 = \frac{4}{5}$

$8 < x \le 9$: $F(x) = F(8) + \int_8^t \frac{1}{5} dt$

$= \frac{4}{5} + \left[\frac{1}{5}t\right]_8^t$

$= \frac{4}{5} + \frac{1}{5}t - \frac{8}{5}$

$= \frac{1}{5}t - \frac{4}{5}$

The cdf is $F(x) = \begin{cases} 0 & t < 0 \\ \frac{1}{80}t^2 & 0 \le t \le 8 \\ \frac{1}{5}t - \frac{4}{5} & 8 < t \le 9 \\ 1 & t > 9 \end{cases}$

(d) $P(T > 7) = 1 - F(7) = 11 - \frac{1}{80} \times 7^2 = \frac{31}{80} = 0.3875$

E.g. 2 A probability density function is given by $f(x) = \begin{cases} kx & 0 \le x \le 2 \\ k(4-x) & 2 < x \le 4 \\ 0 & \text{elsewhere} \end{cases}$, where k is a constant.

(a) Sketch the graph of $f(x)$ and state the mode.
(b) Determine the value of the constant k.
(c) Obtain the cumulative distribution function, $F(x)$.
(d) Calculate the probability $P(1 < X < 3)$.

E.g. 3 The continuous random variable X has cumulative distribution function $F(x)$ where

$F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{12}(2+x) & -2 \le x < 0 \\ \frac{1}{6}(1+x) & 0 \le x < 4 \\ \frac{1}{12}(6+x) & 4 \le x < 6 \\ 1 & x \ge 6 \end{cases}$. Find the pdf of $X, f(x)$, and sketch $y = f(x)$.

E.g. 4 The cumulative distribution function $F(x)$ for a continuous random variable X is defined as

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x - \frac{1}{8}x^2 & 0 \leq x \leq 1 \\ a + \frac{1}{4}x & 1 \leq x \leq 2. \\ b + \frac{1}{8}x^2 - \frac{1}{4}x & 2 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

(a) Find a and b.

(b) Find the pdf, $f(x)$.

Video: Piecewise probability density functions

Solutions to Starter and E.g.s

Exercise

p134 7E Qu 1-4, (5-6 red)

Summary

Piecewise functions have different functions defined for different parts of the domain. When a pdf is defined piecewise, the resulting cdf must still be a continuous function. The usual rules still apply but area calculations require ***more than one definite integral*** to be considered.