Review of AS Vectors

Starter

1. (Review of AS material) Given that $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ -10 \\ 12 \end{pmatrix}$, find:
   (a) $\mathbf{a} \cdot \mathbf{b}$
   (b) the angle between $\mathbf{a}$ and $\mathbf{b}$.

2. (Review of AS material) Convert the line $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ to Cartesian form.

3. (Review of AS material) Convert the line $\dfrac{x+7}{8} = \dfrac{4-y}{3} = 3(z+2)$ to vector form.

4. (Review of AS material) Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. Find:
   (a) $\mathbf{u} \times \mathbf{v}$
   (b) a vector perpendicular to both **u** and **v**

Notes

Vector equation of the line

The vector equation of the line is of the form:

$\mathbf{r} = \mathbf{p} + \lambda \mathbf{d}$ where $\mathbf{p}$ is the position vector of any fixed point on the line,
λ is a scalar ($\lambda \in \mathbb{R}$)
and $\mathbf{d}$ is any direction vector parallel to the line

Scalar product

Definition: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between the vectors.

Component form of scalar product: $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

Angle between vectors and lines: $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

Test for perpendicularity: $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow$ vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to one another

Vector product

Definition: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the ***unit vector perpendicular*** to both $\mathbf{a}$ and $\mathbf{b}$ and θ is the angle between the vectors.

Component form of vector product: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

The vector $\mathbf{a} \times \mathbf{b}$ is ***perpendicular to both*** $\mathbf{a}$ ***and*** $\mathbf{b}$.

Therefore: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{b} \cdot (\mathbf{b} \times \mathbf{a}) = 0$

Useful results: $\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$

$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

$$\mathbf{a}\,.\,(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,.\,(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\,.\,(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Distributive rule: $(\mathbf{p}+\mathbf{q})\times\mathbf{r} = \mathbf{p}\times\mathbf{r}+\mathbf{q}\times\mathbf{r}$

Equation of straight line in 3-D

Vector equation of line

$\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$ where $\mathbf{p}$ is the ***position vector of an fixed point*** on the line
$\mathbf{d}$ is the ***direction vector of a vector parallel*** to the line and
λ is a parameter

Cartesian equation of a line in 3-D

$\dfrac{x-p_1}{d_1} = \dfrac{y-p_2}{d_2} = \dfrac{z-p_3}{d_3} (= \lambda)$ passes through the point (p_1, p_2, p_3) and is ***parallel to the direction vector*** $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

Determining whether lines are parallel, coincident, intersecting or skew

Possibilities for two lines in 2-dimensions: parallel, coincident, intersect.
Possibilities for two lines in 3-dimensions: parallel, coincident, intersect or skew.

Coincident — one line lies on top of the other line i.e. they are the ***same line***.
Skew (3-D only) — lines are ***not parallel and do not intersect***.

Success criteria:

1. By inspection decide whether the lines are parallel to one another:
   Direction vectors are multiple ⇒ parallel or coincident
   Direction vectors are not multiples ⇒ intersecting or skew

If parallel or coincident: see if the fixed point of one line lies on the other line

2. Put fixed point equal to the line and solve the 3 equations for λ:
   All 3 values the same ⇒ the equations are consistent ⇒ coincident (same line)
   All 3 values not the same ⇒ the equations are inconsistent ⇒ (just) parallel

If intersecting or skew:

2. Put the two equations equal to each other and form 3 equations.
3. Solve two of the equations simultaneously to get values for the unknowns.
4. Substitute the values for the unknowns back into the third equation:
   3rd equation is consistent ⇒ intersecting
   3rd equation is consistent ⇒ skew
5. To find the point of intersection, substitute an unknown into its equation.

E.g. 1 Lines L_1, L_2 and L_3 have the following vector equations:

$L_1 : \mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
$L_2 : \mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k})$
$L_3 : \mathbf{r} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \alpha(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$

(a) Show that L_1 and L_2 are parallel but not coincident.
(b) Show that L_1 and L_3 intersect, and find the position vector of their point of intersection.
(c) Show that L_2 and L_3 are skew.

Working: (a) $3(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$ i.e. the direction vectors of L_1 and L_2 are multiples so the lines are parallel.
Does the fixed point of L_2 lie on L_1?
$2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
Equating coefficients:

$\mathbf{i}$: $2 + \lambda = 3 \Rightarrow \lambda = 1$
$\mathbf{j}$: $1 - 2\lambda = 1 \Rightarrow \lambda = 1$
$\mathbf{k}$: $1 + 2\lambda = 2 \Rightarrow \lambda = 0.5$

Since the $\lambda-$values are not all equal, the fixed point of L_2 does not lie on L_1.
Therefore, L_1 and L_2 are not coincident, they are just parallel.

Video: Scalar product
Video: Perpendicular vectors
Video: Vector equation of line
Video: Angle between two lines
Video: Parallel lines
Video: Intersecting and skew lines
Video: Closest point to a line and shortest distance from the origin
Video: Cartesian form of a line
Video: Vector product (cross product)

Exam questions: Scalar product
Exam questions: Vectors

Solutions to Starter and E.g.s

Exercise
See worksheet

Summary
Vector equation of the line in 2- or 3-D: $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$
Scalar product: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between the vectors.

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Angle between vectors and lines: $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

Test for perpendicularity: $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow$ vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to one another

Vector product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The vector $\mathbf{a} \times \mathbf{b}$ is ***perpendicular to both*** $\mathbf{a}$ ***and*** $\mathbf{b}$.

Cartesian equation of a line in 3-D: $$\frac{x - p_1}{d_1} = \frac{y - p_2}{d_2} = \frac{z - p_3}{d_3} (= \lambda)$$

Success criteria — determining whether lines are parallel, coincident, intersecting or skew

1. By inspection decide whether the lines are parallel to one another:
   Direction vectors are multiple $\Rightarrow$ parallel or coincident
   Direction vectors are not multiples $\Rightarrow$ intersecting or skew

If parallel or coincident: see if the fixed point of one line lies on the other line

2. Put fixed point equal to the line and solve the 3 equations for λ:
   All 3 values the same $\Rightarrow$ the equations are consistent $\Rightarrow$ coincident (same line)
   All 3 values not the same $\Rightarrow$ the equations are inconsistent $\Rightarrow$ (just) parallel

If intersecting or skew:

3. Put the two equations equal to each other and form 3 equations.
4. Solve two of the equations simultaneously to get values for the unknowns.
5. Substitute the values for the unknowns back into the third equation:
   3rd equation is consistent $\Rightarrow$ intersecting
   3rd equation is consistent $\Rightarrow$ skew
6. To find the point of intersection, substitute an unknown into its equation.