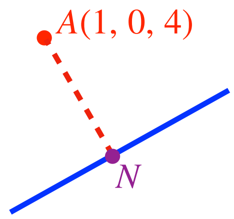
Shortest distances from a point to a line

Starter

1. (Review of last lesson) Determine whether the points $(1, -2, 1)$ and $(-2, 1, 3)$ are on the same or opposite sides of the plane $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 1$.

2. (Review of last lesson) Find the exact values of the coordinates of the point $(5, -3, 8)$ after it has been reflected in the plane given by $x - 2y + 5z + 39 = 0$.

3. Using the simplified diagram to help you, find the shortest distance between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and the point $A(1, 0, 4)$.



Notes

Success criteria 1 — shortest distance from a point, A, to the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$

This method finds the value of λ for the point, N, on the line such that $\overrightarrow{AN}$ is perpendicular to the line.

1. Express the coordinates of the line as a ***column vector***.
2. Find $\overrightarrow{AN}$.
3. Solve the equation $\overrightarrow{AN} \cdot \mathbf{d} = 0$ for λ.
4. Substitute the $\lambda-$value into the equation for $\overrightarrow{AN}$.
5. The shortest distance is equal to $\left|\overrightarrow{AN}\right|$.

E.g. 1 Find the exact value of the perpendicular distance from the point $A(2, 1, 4)$ to the straight line with equation $\dfrac{x-1}{3} = \dfrac{y+2}{5} = z - 1$.

Working: $\dfrac{x-1}{3} = \dfrac{y+2}{5} = z - 1 \quad \Rightarrow \quad \begin{pmatrix} 1+3\lambda \\ -2+5\lambda \\ 1+\lambda \end{pmatrix}$

$$\overrightarrow{AN} = \begin{pmatrix} 1+3\lambda \\ -2+5\lambda \\ 1+\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1+3\lambda \\ -3+5\lambda \\ -3+\lambda \end{pmatrix}$$

Since $\overrightarrow{AN}$ is $\perp$ to the line, $\overrightarrow{AN} \cdot \mathbf{d} = 0$: $\begin{pmatrix} -1+3\lambda \\ -3+5\lambda \\ -3+\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$

$$3(-1+3\lambda) + 5(-3+5\lambda) + -3 + \lambda = 0$$
$$-21 + 35\lambda = 0$$
$$\lambda = \frac{3}{5}$$

When $\lambda = \dfrac{3}{5}$, $\overrightarrow{AN} = \begin{pmatrix} 0.8 \\ 0 \\ -2.4 \end{pmatrix}$.

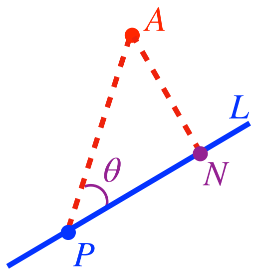
Shortest distance $= \left|\overrightarrow{AN}\right| = \sqrt{0.8^2 + 0^2 + (-2.4)^2} = \dfrac{4\sqrt{10}}{5} \approx 2.53$

Success criteria 2 — shortest distance from a point, A, to the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$

In 3-dimensions, the vector resolute method is another method that can be used, though it is not necessarily quicker than the previous method.

E.g. 2 By considering the relationship between AP, AN and θ, find the shortest distance between the point A, whose position vector is $\mathbf{a}$, and the line $L:\ \mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$.

Hint: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$



Working: Let N be such that AN is perpendicular to the line

i.e. $\overrightarrow{AN}.\mathbf{d} = 0$

The distance required is $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{AP}\right|\sin\theta$.

Multiply both sides by $|\mathbf{d}|$: $\left|\overrightarrow{AN}\right||\mathbf{d}| = \left|\overrightarrow{AP}\right||\mathbf{d}|\sin\theta$

But $\left|\overrightarrow{AP}\right||\mathbf{d}|\sin\theta = \left|\overrightarrow{AP} \times \mathbf{d}\right|$ and $\left|\overrightarrow{AP}\right| = \mathbf{p} - \mathbf{a}$

So AN I **d** I = I (**p** — **a**) × **d** I $\left|\overrightarrow{AN}\right||\mathbf{d}| = \left|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}\right|$

Therefore $\therefore \left|\overrightarrow{AN}\right| = \dfrac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$

In general, the shortest distance from a point with position vector $\mathbf{a}$ to the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$ is:

$$\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$

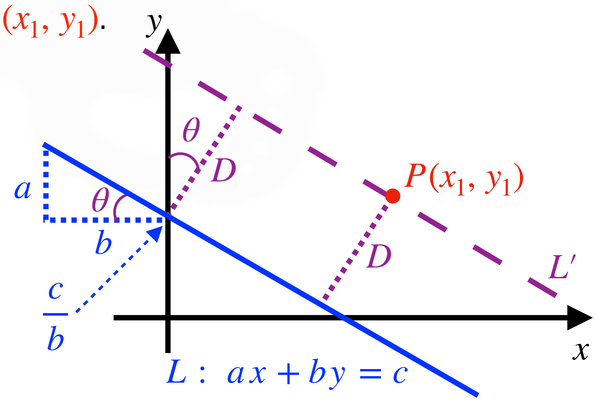
E.g. 3 Find the perpendicular distance of the point $(2, 3, 4)$ from the line whose equation is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 15 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -12 \\ -3 \end{pmatrix}.$$

Finding the shortest distance between a point and a line in 2-dimensions

In 2-dimensions, the vector resolute method doesn't work since the vector product of two vectors cannot be done. A formula can still be derived, though it is easier to do in Cartesian form.

Consider the line $L:\ ax + by = c$ and the point $P(x_1, y_1)$.



To find the shortest, or perpendicular, distance from the point to the line, a line parallel to L is drawn passing through P. Let this line be L'.

The required distance, D, is now in a right-angle triangle whose hypotenuse is on the $y-$axis

To find the length of the hypotenuse, we need the $y-$intercepts of L and L'.

For L: when $x = 0$, $by = c$ so the $y-$intercept is $\dfrac{c}{b}$.

The equation of L' is of the form $ax + by = C$.

Since it passes through $P(x_1, y_1)$: $ax_1 + by_1 = C$ so the equation of the line is $ax + by = ax_1 + by_1$

The $y-$intercept of this line is $\dfrac{ax_1 + by_1}{b}$

So the hypotenuse is $\dfrac{ax_1 + by_1}{b} - \dfrac{c}{b} = \dfrac{ax_1 + by_1 - c}{b}$

Using right-angled trigonometry, the required distance, D, is $\dfrac{ax_1 + by_1 - c}{b} \cos\theta$

The line has gradient $-\dfrac{a}{b}$ so $\tan\theta = \dfrac{a}{b} \quad \Rightarrow \quad \cos\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$

$$D = \frac{ax_1 + by_1 - c}{b} \times \frac{b}{\sqrt{a^2 + b^2}} = \frac{ax_1 + by_1 - c}{\sqrt{a^2 + b^2}}$$

$$D = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$$

Formula booket

The distance between a point and a line is $D = \dfrac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$, where the coordinates of the point are (x_1, y_1) and the equation of the line is given by $ax + by = c$

E.g. 4 Find the exact value of the shortest distance from the point $(6, 2)$ to the line with equation $5x - 3y + 4 = 0$.

Video: **Shortest distance of a point to a line**

Exam questions: **Vectors**

Solutions to Starter and E.g.s

Exercise
p93 4D Qu 2i, 8, 10, 16

Summary

2:D: the shortest distance from the point (x_1, y_1) to the line $ax + by = c$ is:

$$\frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$$

3-D: the shortest distance from a point with position vector $\mathbf{a}$ to the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$ is:

$$\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$$