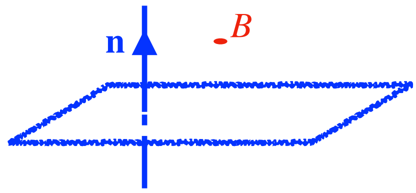
Shortest distance from a point to a plane

Starter

1. **(Review of last lesson)**
Find the angle between the planes $x + 2y - 3z = 1$ and $2x - 4y + z = 3$.

2. Using the simplified diagram on the right, find the exact value shortest distance from the point $B(4, 5, 6)$ to the plane $x + 2y - 2z = 9$.



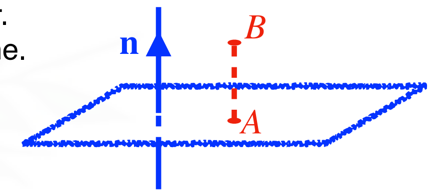
Notes

While the method to the find the shortest distance between a point to the plane in question 2 from the starter is correct (given below), it is not fastest method.

Success criteria 1 — find the value of λ that would give the coordinates of A

N.B. Let A be the point on the plane such that BA is perpendicular to the plane.

1. Find the equation of the line through B and perpendicular to the plane.
2. Express the coordinates of this line as a column vector.
3. Substitute the coordinates into the equation of the plane.
4. Solve for λ.
5. Substitute the value of λ to find the coordinates of the point, A.
6. Find $|\overrightarrow{BA}|$.



N.B. $\overrightarrow{BA} = \lambda\mathbf{n}$ so $|\overrightarrow{BA}| = |\lambda\mathbf{n}|$ i.e. there is no need to find the coordinates of A.

E.g. 1 Use the method above to find the perpendicular distance from the origin to the plane $\mathbf{r}\,.\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = k$ in terms of k, a, b and c.

Working: The equation of the line through the origin and $\perp$ to the plane is $\mathbf{r} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$.

Substituting into $ax + by + cz = k$: $\quad \lambda a^2 + \lambda b^2 + \lambda c^2 = k$

$$\therefore \lambda = \frac{k}{a^2 + b^2 + c^2}$$

$$\overrightarrow{ON} = \lambda\mathbf{n} = \frac{k}{a^2 + b^2 + c^2} \times (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

$$\left|\overrightarrow{ON}\right| = \frac{k}{a^2 + b^2 + c^2} \times \sqrt{a^2 + b^2 + c^2} = \frac{k}{\sqrt{a^2 + b^2 + c^2}}$$

The distance from the plane to the origin is $\dfrac{k}{\sqrt{a^2 + b^2 + c^2}}$.

N.B. $\sqrt{a^2 + b^2 + c^2} = |\mathbf{n}|$

In general, the displacement to the origin from the plane $\mathbf{r}\,.\,\mathbf{n} = k$ is $\dfrac{k}{|\mathbf{n}|}$

N.B. This value can be positive or negative depending on the value of k i.e. depending on which side of the origin the point is.

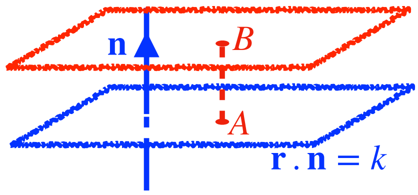
The form $\mathbf{r}\,.\,\hat{\mathbf{n}} = \dfrac{k}{|\mathbf{n}|}$ gives the distance to the origin.

E.g. 2 Find the perpendicular distance from the origin to the following planes:
(a) $\mathbf{r}\,.\,(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 57$ (b) $\mathbf{r}\,.\,(3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 195$

Working: (a) Distance $= \dfrac{k}{|\mathbf{n}|} = \dfrac{57}{\sqrt{1^2 + 2^2 + (-2)^2}} = 19$

How can the above be used to find the distance between the point A to the plane $\mathbf{r}\,.\,\mathbf{n} = k$?

E.g. 3 Let the point B have position vector $\mathbf{b}$.
By finding the equation of the plane parallel to $\mathbf{r}\,.\,\mathbf{n} = k$ that passes through A and using the distances of the planes to the origin, find the shortest distance from the point B to the plane $\mathbf{r}\,.\,\mathbf{n} = k$.



Working: The plane parallel to $\mathbf{r}\,.\,\mathbf{n} = k$ that passes through A is $\mathbf{r}\,.\,\mathbf{n} = \mathbf{b}\,.\,\mathbf{n}$

The displacement from $\mathbf{r}\,.\,\mathbf{n} = k$ to the origin is $\dfrac{k}{|\mathbf{n}|}$

The displacement from $\mathbf{r}\,.\,\mathbf{n} = \mathbf{b}\,.\,\mathbf{n}$ to the origin is $\dfrac{\mathbf{b}\,.\,\mathbf{n}}{|\mathbf{n}|}$

The shortest displacement between the planes is $\dfrac{\mathbf{b}\,.\,\mathbf{n}}{|\mathbf{n}|} - \dfrac{k}{|\mathbf{n}|} = \dfrac{\mathbf{b}\,.\,\mathbf{n} - k}{|\mathbf{n}|}$

The shortest ***distance*** from the point to the plane is $\dfrac{|\mathbf{b}\,.\,\mathbf{n} - k|}{|\mathbf{n}|}$.

Success criteria 2 — shortest distance from a point to the a plane (formula)
The shortest ***distance*** from the point B, with position vector $\mathbf{b}$, to the plane is $\mathbf{r}\,.\,\mathbf{n} = k$ is:

$$\frac{|\mathbf{b}\,.\,\mathbf{n} - k|}{|\mathbf{n}|}$$

Formula booklet

The distance between a point and a plane is $D = \dfrac{|\mathbf{b}.\mathbf{n} - p|}{|\mathbf{n}|}$, where $\mathbf{b}$ is the position vector of the point and the equation of the plane is given by $\mathbf{r}.\mathbf{n} = p$

E.g. 4 Find the shortest distance from the point $B(25,\ 5,\ 7)$ to the plane $12x + 4y + 3z = 3$.

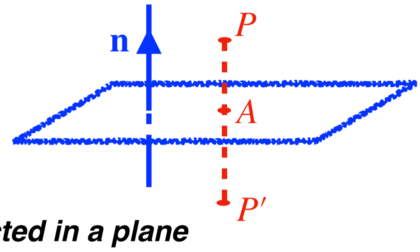
E.g. 5 Find a formula to find the shortest distance from the point $B(X,\ Y,\ Z)$ to the plane $\mathbf{r}\,.\,(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = k$

In the formula booklet, the equation for 2-dimensions appears:

The distance between a point and a line is $D = \dfrac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$, where the coordinates of the point are (x_1, y_1) and the equation of the line is given by $ax + by = c$

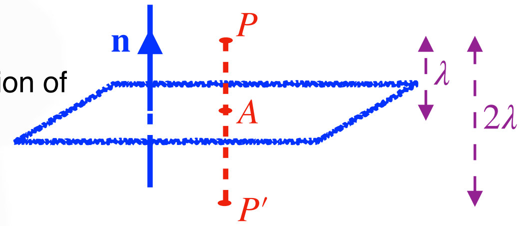
Points reflected in planes

E.g. 6 Find the reflection of the point $P(5,\ 7,\ 11)$ in the plane $2x + 3y + 5z = 10$.



Success criteria — finding the coordinates of a point, P, reflected in a plane

1. Find the equation of the line passing through P and perpendicular to the plane.
2. Express the coordinates of this line as a column vector.
3. Substitute the coordinates into the equation of the plane.
4. Solve for λ.
5. Double the value of λ.
6. Substitute the doubled $\lambda-$value into the equation of the line.



E.g. 7 Find the image of the origin following reflection in the plane $x + 2y + 3z = 14$

Video: Shortest distance from a point to a plane

Solutions to Starter and E.g.s

Exercise

p93 4D Qu 1, 6, 7, 12, 13

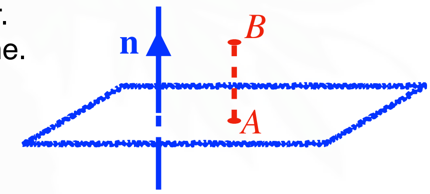
Summary

Finding the shortest distance between a point and a plane:

Success criteria 1 — find the value of λ that would give the coordinates of A:

N.B. Let A is the point on the plane such that BA is perpendicular to the plane.

1. Find the equation of the line through A and perpendicular to the plane.
2. Express the coordinates of this line as a column vector.
3. Substitute the coordinates into the equation of the plane.
4. Solve for λ.
5. Find $|\overrightarrow{BA}|$.



Success criteria 2 — use a formula

The shortest ***distance*** from the point B, with position vector $\mathbf{b}$, to the plane is $\mathbf{r} \,.\, \mathbf{n} = k$ is:

$$\frac{|\mathbf{b} \,.\, \mathbf{n} - k|}{|\mathbf{n}|}$$

The shortest distance from the point $P(X,\ Y,\ Z)$ to the plane $\mathbf{r} \,.\, (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = k$ is:

$$\frac{\left|aX + bY + cZ - k\right|}{\sqrt{a^2 + b^2 + c^2}}$$

Success criteria — finding the coordinates of a point, P, reflected in a plane:

1. Find the equation of the line passing through P and perpendicular to the plane.
2. Express the coordinates of this line as a column vector.
3. Substitute the coordinates into the equation of the plane.
4. Solve for λ.
5. Double the value of λ.
6. Substitute the doubled $\lambda-$value into the equation of the line.