Simple harmonic motion

Starter

1. Find the general solution of the equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$.

Notes

Simple harmonic motion (***SHM***) is when the ***acceleration*** of a particle is proportional to its displacement but acts in the opposite direction i.e. $\frac{d^2x}{dt^2} \propto -x$

So SHM can be modelled by $\frac{d^2x}{dt^2} = -\omega^2 x$ or $\ddot{x} = -\omega^2 x$ and its general solution is $x = A\sin\omega t + B\cos\omega t$, where ω is called the ***angular frequency***, whose units are s^{-1}.

The force acting is always acting to return the object to the centre position and the maximum velocity occurs when the object passes through the centre position. The average position around which the object oscillates is called the ***centre line***.

SHM is used to describe motion with ***small oscillations***, for example with ***springs or pendulums***, and so it would be useful to know the amplitude and period of oscillations. To do this we need to write it in a form involving only one trigonometric ratio.

Geogebra 1: SHM spring
Geogebra 2: SHM spring
SHM animation

E.g. 1 Express $x = A\sin\omega t + B\cos\omega t$ in the form $x = R\sin(\omega t + \phi)$. Hence state the amplitude and period of the oscillations

Initial position

E.g. 2 The general solution for SHM is $x = A\sin\omega t + B\cos\omega t$.

(a) Given that the initial position of the object is on the centre line, find the equation of the general solution and state the amplitude of the motion.

(b) Instead, if initially the object is at the maximum displacement, find the equation of the general solution and state the amplitude of the motion.

E.g. 3 Consider the differential equation $\frac{d^2x}{dt^2} + 9x = 0$.

(a) Write down the general solution.

(b) Given the initial conditions $x = 0$ and $\frac{dx}{dt} = 1$, find the particular solution.

(c) Write down the period and amplitude of the oscillations and sketch a graph.

Connecting velocity and displacement

E.g. 4 Given that $a = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx}$ derive a formula for velocity, v, in terms of x and a, where a is the amplitude of the SHM.

N.B. The relationship between velocity and displacement for a particle moving with SHM is $v^2 = \omega^2(a^2 - x^2)$.

E.g. 5 A particle's motion is described by the differential equation $\frac{d^2x}{dt^2} + 16x = 0$. Its amplitude is 0.3 m.

(a) State the angular frequency of the motion.
(b) Find the maximum velocity of the particle.
(c) Find the velocity of the particle when it is half-way between the centre position and its maximum displacement.

Limitations:

- The formula only works for small oscillations (20° or less).
- The formula suggest the motion will continue forever but air resistance will decrease oscillations over time such that motion will cease.
- The mass of the string is neglected.

Video 1: Simple harmonic motion
Video 2: Simple harmonic motion

Solutions to Starter and E.g.s

Exercise
p248 11B Qu 1i, 2i, 3i, 4-12

Summary

The differential equation for simple harmonic motion is $\frac{d^2x}{dt^2} + \omega^2 x = 0$.

The general solution for SHM is $x = A\sin\omega t + B\cos\omega t$, where ω is called the ***angular frequency*** of the motion, whose units are s^{-1}.

Amplitude: $R = \sqrt{A^2 + B^2}$

Period: $T = \frac{2\pi}{\omega}$

If the initial position is the ***centre line***: $x = A\sin\omega t$
If the initial position is the ***maximum displacement***: $x = B\cos\omega t$
Relationship between ***velocity*** and ***displacement***: $v^2 = \omega^2(a^2 - x^2)$