Using Integrating Factors to Solve Differential Equations

Starter

1. (Review of last lesson) Solve $-\operatorname{cosec}^2 x e^{\cosh y} + \cot x \sinh y e^{\cosh y}\frac{dy}{dx} = \sec x \tan x$.

2. Look at these differential equations and decide what we need to do to put them in the exact form: $f(x)g'(y)\frac{dy}{dx} + f'(x)g(y) = Q(x)$. Hence, solve the equations.

   Hint: what do you need to multiply the equation by?

   (a) $x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$ (b) $xy\frac{dy}{dx} + y^2 = 3x$ (c) $\frac{dy}{dx} + 3y = e^{-3x}$

Working: (a)

$$x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$$

Multiply by x^2:

$$x^3\frac{dy}{dx} + 3x^2y = e^x$$

$$\frac{d(x^3y)}{dx} = e^x$$

$$x^3y = \int e^x dx$$

$$x^3y = e^x + c$$

Notes

When a linear first order differential equation is not in exact form, an ***integrating factor*** can be used to get it into the form needed prior to integration.

Consider differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Assume the right-hand side of the equation can be turned into the correct form by multiplying the equation by $I(x)$, where $I(x)$ is the ***integrating factor***.

So $\quad I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x) \quad \equiv \quad f(x)\frac{dy}{dx} + f'(x)y = \ldots$

For the assumption to be true: $\quad f(x) = I(x) \quad$ and $\quad f'(x) = I(x)P(x)$.

Since $f(x) = I(x) \quad \Rightarrow \quad f'(x) = I'(x)$

Substitute into the 2nd equation: $\quad I'(x) = I(x)P(x) \quad \Rightarrow \quad \frac{dI}{dx} = I(x)P(x)$.

By separation of variables:

$$\int \frac{1}{I(x)} dI = \int P(x)dx$$

$$\ln I(x) = \int P(x)dx$$

$$I(x) = e^{\int P(x)dx}$$

is the ***integrating factor***

N.B. The integrating factor only works for differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$ so make sure the equation is in this form before finding $I(x)$.

E.g. 1 Solve these differential equations:

(a) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{e^x}{x}$ (b) $\dfrac{dy}{dx} + 3y = e^{-3x}$ (c) $x\dfrac{dy}{dx} + 2y = e^{x^2}$

Working: (a) $I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiply by x:

$$x\frac{dy}{dx} + y = e^x$$

$$\frac{d(xy)}{dx} = e^x$$

$$xy = \int e^x dx$$

$$xy = e^x + c$$

N.B. $e^{\ln x} = x$ so $e^{k\ln x} = x^k$

E.g. 2 Find the particular solution of the differential equation $(x+1)\dfrac{dy}{dx} - 3y = (x+1)^4$ which satisfies the conditions that $y = 16$ when $x = 1$

Video: Solving differential equations using an integrating factor

Exam questions: Integrating factors

Solutions to Starter and E.g.s

Exercise
p230 10B Qu 1i, 2-6, 8

Summary

When a differential equations is in the form $\dfrac{dy}{dx} + P(x)y = Q(x)$, the equation can be multiplied by the integrating factor $I(x) = e^{\int P(x)dx}$ so that it is in exact form.