Vector form of work done, kinetic energy and power

Starter

(Review of AS Ma Pure material)

The definition of the scalar product is: $\mathbf{a} \cdot \mathbf{b} = |a||b|\cos\theta$.

Component form of scalar product: $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 \times v_1 + u_2 \times v_2 + u_3 \times v_3$

N.B. When writing by hand remember to write the tilde (or squiggle) under the vectors i.e. $\underset{\sim}{u}$.

1. Let $\mathbf{p} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 3 \\ 9 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -5 \\ -7 \end{pmatrix}$. Find:

   (a) $\mathbf{p} \cdot \mathbf{q}$ (b) $\mathbf{q} \cdot \mathbf{r}$ (c) the angle between $\mathbf{p}$ and $\mathbf{r}$.

Notes

Using quantities in 1-D, Work done = Force × distance

Kinetic energy, KE $= \frac{1}{2}mv^2$

When using vectors in 2-D and 3-D, we multiply the vectors together using the scalar product.

Work done in vector form

Let the force $\mathbf{F}$ be acting at θ to the line of displacement, $\mathbf{s}$.

Work done $= |\mathbf{F}|\cos\theta \times |\mathbf{s}|$

Work done $= \mathbf{F} \cdot \mathbf{s}$

i.e. ***work done*** equals the ***scalar product*** of the ***force*** and the ***displacement vectors***.

E.g. 1 Calculate the work done by a force $\mathbf{F} = (3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$ N when it acts over a displacement $\mathbf{s} = (2\mathbf{i} - \mathbf{j} - 5\mathbf{k})$ m.

Kinetic energy in vector form

In 1-D, kinetic energy, KE $= \frac{1}{2}mv^2$.

When v becomes the vector $\mathbf{v} = (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$ m/s, we know that $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

So $|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$ …but this equals $\mathbf{v} \cdot \mathbf{v}$.

Therefore, kinetic energy: KE $= \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}$

Connecting work done and change in kinetic energy

The work-energy principle states that the change in kinetic energy equals the work done (assuming gravitational potential energy does not change).

Let the initial and final velocities be $\mathbf{u}$ and $\mathbf{v}$ respectively.

$$\mathbf{F} \cdot \mathbf{s} = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} - \frac{1}{2}m\mathbf{u} \cdot \mathbf{u}$$

SUVAT equations in vector form

Scalar form	Vector form	Unknown not included
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$	No $\mathbf{s}$
$s = \frac{1}{2}(u + v)t$	$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$	No $\mathbf{a}$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$	No $\mathbf{v}$
$s = vt - \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$	No $\mathbf{u}$
$v^2 = u^2 + 2as$	$\mathbf{v}\,.\,\mathbf{v} = \mathbf{u}\,.\,\mathbf{u} + 2\mathbf{a}\,.\,\mathbf{s}$	No t

In addition: $\mathbf{F} = m\mathbf{a}$

E.g. 2 A particle of mass 5 kg moving with kinetic energy 15 J has an acceleration $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ as it moves through a displacement $\mathbf{s} = 2\mathbf{j} + 5\mathbf{k}$. Find its final kinetic energy.

Working: Initial KE $= 15$ so $\frac{1}{2} \times 5 \times \mathbf{u}\,.\,\mathbf{u} = 15 \quad \Rightarrow \quad \mathbf{u}\,.\,\mathbf{u} = 6$

$\mathbf{v}\,.\,\mathbf{v} = \mathbf{u}\,.\,\mathbf{u} + 2\mathbf{a}\,.\,\mathbf{s}$: $\quad \mathbf{v}\,.\,\mathbf{v} = 6 + 2(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\,.\,(2\mathbf{j} + 5\mathbf{k})$

$\mathbf{v}\,.\,\mathbf{v} = 6 + 2 = 8$

Final KE $= \frac{1}{2}m\mathbf{v}\,.\,\mathbf{v} = \frac{1}{2} \times 5 \times 8 = 20$

Its final kinetic energy is 20 J

Power in vector form

From AS level:

$$\text{Average power} = \frac{\text{work done}}{\text{time taken}} = \frac{\text{force} \times \text{distance}}{\text{time taken}} = \text{force} \times \frac{\text{distance}}{\text{time taken}} = \text{force} \times \text{speed}$$

So $P = Fv$

In vector form this becomes: Power $= \mathbf{F}\,.\,\mathbf{v}$

E.g. 3 A motor boat is sailing with constant velocity $(32\mathbf{i} + 21\mathbf{j})$ m/s. The motor is producing a force $(25\mathbf{i} + 20\mathbf{j})$ N. Find the power at which the motor is working.

Video (password needed): **Work done in vector form**

Solutions to Starter and E.g.s

Exercise

p164 6D Qu 1-10

Summary

In vector form:

Work done $= \mathbf{F}\,.\,\mathbf{s}$

Kinetic energy $= \frac{1}{2}m\mathbf{v}\,.\,\mathbf{v}$

Work energy principle: $\mathbf{F}\,.\,\mathbf{s} = \frac{1}{2}m\mathbf{v}\,.\,\mathbf{v} - \frac{1}{2}m\mathbf{u}\,.\,\mathbf{u}$