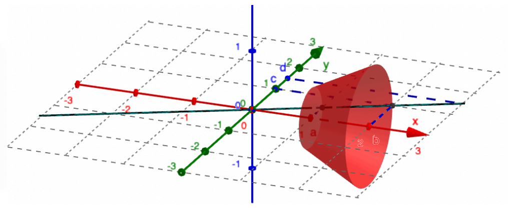
Volumes of revolution

Starter

1. (Review of last lesson) Find $\int_1^{\infty} \frac{x}{1+x^2} dx$ and state whether it is convergent or divergent.

Notes

When a curve is rotated $360°$ around the $x-$axis a 3-D solid is formed. This solid is called the ***volume of revolution*** or ***solid of revolution***

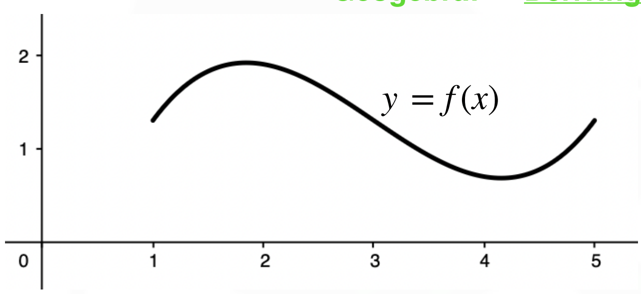


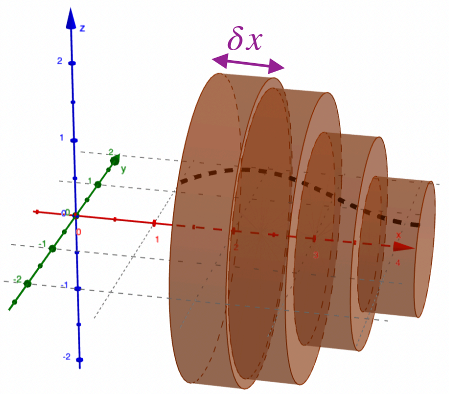
Geogebra: Volume of revolution

Deriving the formula

The derivation of the formula for the volume of solid of revolution is similar to finding the area between a curve and the $x-$axis. In this case, the volume is split up into discs of width δx and height, y or $f(x)$.

Geogebra: Deriving the formula





N.B. $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

Since volume of cylinder $= \pi r^2 h$, each disc has volume δV where $\delta V \approx \pi y^2 \delta x$.

Rearranging gives: $\frac{\delta V}{\delta x} \approx \pi y^2$

$\lim_{\delta x \to 0} \frac{\delta V}{\delta x} = \frac{dV}{dx} \quad \Rightarrow \quad \frac{dV}{dx} = \pi y^2$

So $V = \pi \int_{x_1}^{x_2} y^2 dx$ around the $x-$axis (limits must be $x-$values)

E.g. 1 Find the exact value of the volume of solid formed when the area between the curve $y = 3x^2$ for $0 \le x \le 3$ and the $x-$axis is rotated through $360°$ about the $x-$axis.

Working: $V = \pi \int_0^3 (3x^2)^2 dx = \pi \int_0^3 9x^4 dx = \pi \left[\frac{9}{5}x^5\right]_0^3 = \frac{2187}{5}\pi = 437.4\pi$

E.g. 2 Calculate the volume of the frustum of a cone formed by rotating the area between the line $y = 2x + 3$, the ordinates $x = 1$ and $x = 4$, and the $x-$axis through 2π about the $x-$axis.

Curve rotated about the $y-$axis

E.g. 3 The volume of solid of revolution about the $x-$axis is $V_x = \pi \int_{x_1}^{x_2} y^2 dx$. Write down the equation for volume of the solid which is rotated around the $y-$axis.

Working: $V_y = \pi \int_{y_1}^{y_2} x^2 dy$ (notice that the limits are $y-$values)

E.g. 4 Find the exact value of the volume formed when the curve $y = x^3$ is rotated 2π about the $y-$axis between $y = 1$ and $y = 8$.

Curves defined parametrically

Replace y^2 (or x^2) by the corresponding function of t and write $dx = \frac{dx}{dt}dt$ (or $dy = \frac{dy}{dt}dt$).

i.e. $V_x = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$ or $V_y = \pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt} dt$

E.g. 5 Find the volume obtained when the curve given by the parametric equations $x = t^3 + 1$, $y = t^2$, between $t = 0$ and $t = 2$ is rotated through 2π radians about the $x-$axis.

Video: Volume of revolution (x-axis)
Video: Volume of revolution (y-axis)
Video: Volume of revolution (parametric)

Volume of revolution (x-axis) EQ
Volume of revolution (y-axis) EQ
Volume of revolution (parametric) EQ

Solutions to Starter and E.g.s

Exercise
p186 8D Qu (1i, 2i, 3i, 4, 5, 6,) 7-19

Summary

Volume of solid of revolution about the $x-$axis: $V_x = \pi \int_{x_1}^{x_2} y^2 dx$

Volume of solid of revolution about the $y-$axis: $V_y = \pi \int_{y_1}^{y_2} x^2 dy$

Parametric forms: $V_x = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$ (about $x-$axis)

$V_y = \pi \int_{t_1}^{t_2} x^2 \frac{dy}{dt} dt$ (about $y-$axis)