Write yours and your teacher's name at the top of your answer sheets.

U6 Further Mathematics Mock

Paper 1 (Pure)
February 2022
2021-2022

Duration: 2 hour

Total number of marks: 93

Write your answers on file paper.

You are permitted to use a scientific or graphical calculator in this paper.

Final answers should be given to a degree of accuracy appropriate to the context.

Students need a formula booklet.

1.

The curve C has equation

$$y = \arccos\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2$$

(a) Show that C has no stationary points.

(3)

The normal to C, at the point where $x = 1$, crosses the x-axis at the point A and crosses the y-axis at the point B.

Given that O is the origin,

(b) show that the area of the triangle OAB is $\frac{1}{54}\left(p\sqrt{3} + q\pi + r\sqrt{3}\pi^2\right)$ where p, q and r are integers to be determined.

(5)

2.

The plane Π_1 has equation

$$\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = -8$$

(a) Find the perpendicular distance from the point (8, 2, 10) to Π_1

(3)

The plane Π_2 has equation

$$\mathbf{r} = \lambda(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ is perpendicular to Π_2

(2)

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2

(3)

(d) Find a vector equation of the line of intersection of the planes Π_1 and Π_2

(4)

3.

(i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

4.

Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

(6)

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

(1)

5.

Show that the solutions to the equation

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

can be expressed in the form

$$x = \pm\ln\left(a + \sqrt{b}\right)$$

where a and b are integers to be found.

You may use without proof the result $\cosh^{-1} y = \ln\left(y + \sqrt{y^2 - 1}\right)$

[5 marks]

6.

Solutions based entirely on graphical or numerical methods are not acceptable.

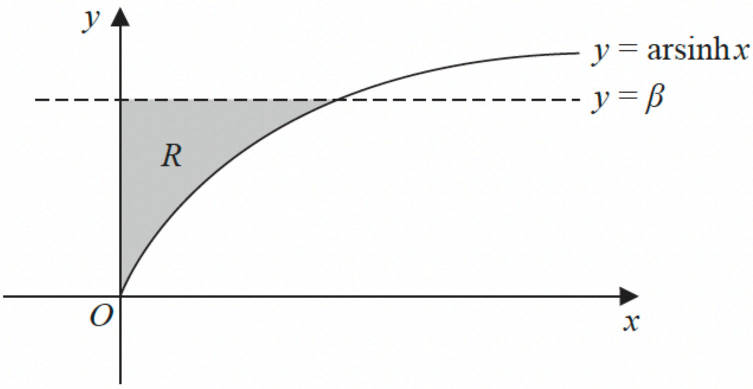

Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \text{arsinh}\, x \qquad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \text{arsinh}\, x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

7.

In this question you must show detailed reasoning.

(a) Find the values of A, B and C for which $\dfrac{x^3+x^2+9x-1}{x^3+x^2+4x+4} \equiv A + \dfrac{Bx+C}{x^3+x^2+4x+4}$. **[1]**

(b) Hence express $\dfrac{x^3+x^2+9x-1}{x^3+x^2+4x+4}$ using partial fractions. **[5]**

(c) Using your answer to part **(b)**, determine $\displaystyle\int_0^2 \frac{x^3+x^2+9x-1}{x^3+x^2+4x+4}\,dx$ expressing your answer in the form $a + \ln b + c\pi$ where a is an integer, and b and c are both rational. **[4]**

8.

The points $A(7, 2, 8)$, $B(7, -4, 0)$ and $C(3, 3.2, 9.6)$ all lie in the plane Π.

(a) Find a Cartesian equation of the plane Π.

[3 marks]

(b) The line L_1 has equation $\mathbf{r} = \begin{bmatrix} 5 \\ -0.4 \\ 4.8 \end{bmatrix} + \mu \begin{bmatrix} 15 \\ 3 \\ 4 \end{bmatrix}$

(b) (i) Show that L_1 lies in the plane Π.

[2 marks]

(b) (ii) Show that every point on L_1 is equidistant from B and C.

[4 marks]

(c) The line L_2 lies in the plane Π, and every point on L_2 is equidistant from A and B.

Find an equation of the line L_2

[4 marks]

(d) The points A, B and C all lie on a circle G.
The point D is the centre of circle G.

Find the coordinates of D.

[3 marks]

9.

Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + 10y - 30$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x + 3y - 4$$

(a) Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)