Write yours and your teacher's name at the top of your answer sheets.

U6 Further Mathematics Mock

Paper 1 (Pure)
February 2023
2022-2023

Duration: 2 hours

Total number of marks: 90

Write your answers on file paper.

You are permitted to use a scientific or graphical calculator in this paper.

Final answers should be given to a degree of accuracy appropriate to the context.

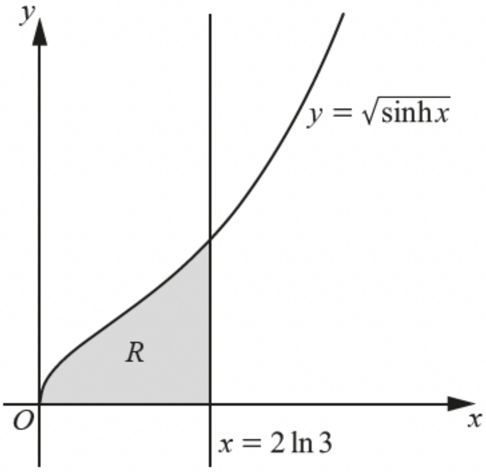
Students need a formula booklet.

1.

In this question you must show detailed reasoning.

(a) Show that $\cosh(2\ln 3) = \frac{41}{9}$. **[2]**

The region R is bounded by the curve with equation $y = \sqrt{\sinh x}$, the x-axis and the line with equation $x = 2\ln 3$ (see diagram). The units of the axes are centimetres.



A manufacturer produces bell-shaped chocolate pieces. Each piece is modelled as being the shape of the solid formed by rotating R completely about the x-axis.

(b) Determine, according to the model, the exact volume of one chocolate piece. **[4]**

2.

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

The plane Π has equation $\mathbf{r}.\begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = 4$.

(a) Find the position vector of the point of intersection of l_1 and Π. **[3]**

(b) Find the acute angle between l_1 and Π. **[3]**

A is the point on l_1 where $\lambda = 1$.

l_2 is the line with the following properties.

- l_2 passes through A
- l_2 is perpendicular to l_1
- l_2 is parallel to Π

(c) Find, in vector form, the equation of l_2. **[3]**

3.

$$y = \cosh^n x \qquad n \geqslant 5$$

(a) (i) Show that

$$\frac{d^2 y}{dx^2} = n^2 \cosh^n x - n(n-1)\cosh^{n-2} x$$

(4)

(ii) Determine an expression for $\frac{d^4 y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form.

(2)

4.

In this question you must show detailed reasoning.

The complex number $-4 + i\sqrt{48}$ is denoted by z.

(a) Determine the cube roots of z, giving the roots in exponential form. **[6]**

The points which represent the cube roots of z are denoted by A, B and C and these form a triangle in an Argand diagram.

(b) Write down the angles that any lines of symmetry of triangle ABC make with the positive real axis, justifying your answer. **[3]**

5.

(a) Express as partial fractions

$$\frac{2x^2 + 3x + 6}{(x+1)(x^2+4)}$$

(3)

(b) Hence, show that

$$\int_0^2 \frac{2x^2 + 3x + 6}{(x+1)(x^2+4)}\, dx = \ln\left(a\sqrt{2}\right) + b\pi$$

where a and b are constants to be determined.

(4)

6.

(i) Evaluate the improper integral

$$\int_1^{\infty} 2e^{-\frac{1}{2}x}\, dx$$

(3)

(ii) The air temperature, θ °C, on a particular day in London is modelled by the equation

$$\theta = 8 - 5\sin\left(\frac{\pi}{12}t\right) - \cos\left(\frac{\pi}{6}t\right) \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

(a) Use calculus to show that the mean air temperature on this day is 8 °C, according to the model.

(3)

Given that the actual mean air temperature recorded on this day was higher than 8 °C,

(b) explain how the model could be refined.

(1)

7.

(a) By using the exponential definitions of $\sinh x$ and $\cosh x$, prove the identity $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$. **[2]**

(b) Hence find an expression for $\cosh 2x$ in terms of $\cosh x$. **[1]**

(c) Determine the solutions of the equation $5\cosh 2x = 16\cosh x + 21$, giving your answers in exact logarithmic form. **[4]**

8.

The coordinates of the points A and B are $(3, -2, -1)$ and $(13, 10, 9)$ respectively.

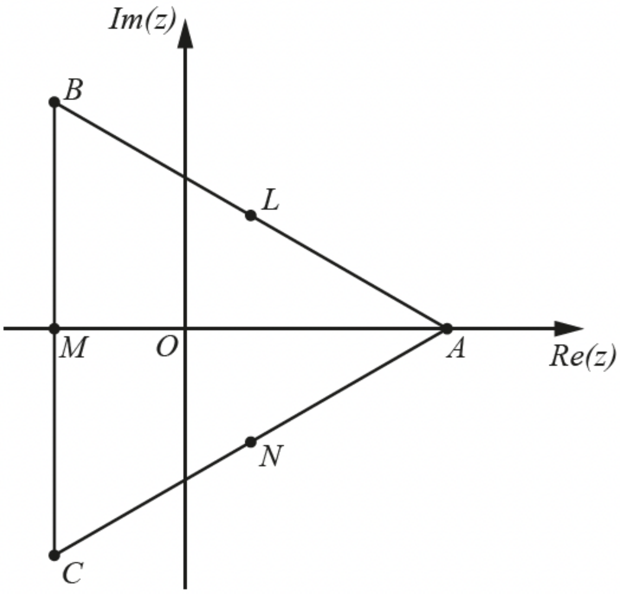
- The plane Π_A contains A and the plane Π_B contains B.
- The planes Π_A and Π_B are parallel.
- The x and y components of any normal to plane Π_A are equal.
- The shortest distance between Π_A and Π_B is 2.

There are **two** possible solution planes for Π_A which satisfy the above conditions.

Determine the acute angle between these two possible solution planes. **[8]**

9.

The cube roots of unity are represented on the Argand diagram below by the points A, B and C.



The points L, M and N are the midpoints of the line segments AB, BC and CA respectively.

Determine a degree 6 polynomial equation with integer coefficients whose roots are the complex numbers represented by the points A, B, C, L, M and N. **[5]**

10.

(a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

11.

(a) Two of the solutions to the equation $\cos 6\theta = 0$ are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

Find the other solutions to the equation $\cos 6\theta = 0$ for $0 \leq \theta \leq \pi$

[2 marks]

(b) Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

[5 marks]

(c) Use the fact that $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ are solutions to the equation $\cos 6\theta = 0$ to find a factor of $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ in the form $(a\cos^2\theta + b)$, where a and b are integers.

[4 marks]

(d) Hence show that

$$\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\frac{2+\sqrt{3}}{4}}$$

[5 marks]