U6 FM Mock Teacher X 22-23 SOLUTIONS [90]

1.

(a)		**DR** $\cosh(2\ln 3) = \frac{e^{2\ln 3}+e^{-2\ln 3}}{2}$	M1	1.1	Correct use of definition of $\cosh x$ must be seen
		$= \frac{1}{2}\left(9+\frac{1}{9}\right) = \frac{41}{9}$	A1	2.1	AG, must see either $e^{\ln 9}$ and $e^{\ln\frac{1}{9}}$ or 3^2 and 3^{-2} or $\frac{1}{2}\left(9+\frac{1}{9}\right)$
			[2]		
(b)		**DR**	M1	3.3	oe, intention to integrate y^2. Condone missing π, ignore limits.
		$V = \pi\int_0^{2\ln 3}\left(\sqrt{\sinh x}\right)^2 dx$ $= \pi[\cosh x]_0^{2\ln 3}$	A1		For $+\cosh x$. Ignore any reference to c
		$= \pi(\cosh(2\ln 3)-\cosh 0)$ $= \pi\left(\frac{41}{9}-1\right)$	M1		Substituting correct limits and subtracting
		$= \frac{32}{9}\pi$ (cm^3) oe	A1		Ignore units
			[4]		

2.

(a)		$\left(\begin{pmatrix}1\\-3\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\-2\end{pmatrix}\right).\begin{pmatrix}2\\-5\\-3\end{pmatrix}=4$	M1	1.1	Substituting the expression for a point on the line into the equation of the plane	
		$2+15-9+\lambda(6-10+6)=4$ $8+2\lambda=4 \Rightarrow 2\lambda=-4 \Rightarrow \lambda=-2$ so	M1	1.1	Dotting out to form and solve equation in λ	
		$\mathbf{r}=\begin{pmatrix}1\\-3\\3\end{pmatrix}+-2\begin{pmatrix}3\\2\\-2\end{pmatrix}=\begin{pmatrix}-5\\-7\\7\end{pmatrix}$	A1	1.1		Condone coordinates
			[3]			
(b)		$\frac{\begin{pmatrix}2\\-5\\-3\end{pmatrix}.\begin{pmatrix}3\\2\\-2\end{pmatrix}}{\sqrt{4+25+9}\sqrt{9+4+4}}$ soi $=\frac{6-10+6}{\sqrt{38}\sqrt{17}}=\frac{2}{\sqrt{646}}=0.07868...$	M1	1.1	BC. Using $\cos\theta=\frac{\mathbf{a.b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	May see $\sin\phi=\frac{\mathbf{a.b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ Or use of cross product
		$\theta=$ awrt $85.5...°$ soi	A1	1.1	Can be implied by correct final answer	or 1.49… rads
		$(\phi=90°-85.48...°=)$ awrt $4.51°$	A1	1.1		or 0.0788 rads
			[3]			
(c)		$\lambda=1\Rightarrow\mathbf{r}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$	B1	3.1a		
		$\mathbf{b}=\begin{pmatrix}3\\2\\-2\end{pmatrix}\times\begin{pmatrix}2\\-5\\-3\end{pmatrix}=\begin{pmatrix}-16\\5\\-19\end{pmatrix}$	M1	2.2a	Method shown or at least two terms correctly evaluated	
		So equation of l_2 is $\mathbf{r}=\begin{pmatrix}4\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}-16\\5\\-19\end{pmatrix}$ oe	A1	1.1	Must be $\mathbf{r}=$. Allow parameter λ.	
			[3]			

3.

(a)(i)	$\frac{dy}{dx} = \ldots \cosh^{n-1} x \sinh x$ $$\frac{d^2y}{dx^2} = \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^{n-1} x \cosh x$$ **Alternatively** $y = \left(\frac{e^x+e^{-x}}{2}\right)^n$ leading to $\frac{dy}{dx} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-1} \left(\frac{e^x-e^{-x}}{2}\right)$ $$\frac{d^2y}{dx^2} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-2} \left(\frac{e^x-e^{-x}}{2}\right)^2 + \ldots \left(\frac{e^x+e^{-x}}{2}\right)^n$$	M1	1.1b
	$$\frac{dy}{dx} = n \cosh^{n-1} x \sinh x$$ $$\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x \sinh^2 x + n \cosh^n x$$ **Alternatively** $$\frac{dy}{dx} = n \left(\frac{e^x+e^{-x}}{2}\right)^{n-1} \left(\frac{e^x-e^{-x}}{2}\right)$$ $$\frac{d^2y}{dx^2} = n(n-1) \left(\frac{e^x+e^{-x}}{2}\right)^{n-2} \left(\frac{e^x-e^{-x}}{2}\right)^2 + n \left(\frac{e^x+e^{-x}}{2}\right)^n$$	A1	2.1
	$$\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x (\cosh^2 x - 1) + n \cosh^n x$$	M1	2.1
	$\frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x$* cso	A1*	1.1b
		(4)	
(a)(ii)	$$\frac{d^3y}{dx^3} = \ldots \cosh^{n-1} x \sinh x - \ldots \cosh^{n-3} x \sinh x$$ $\frac{d^4y}{dx^4}$ $= \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^n x - \ldots \cosh^{n-4} x \sinh^2 x - \ldots cos$	M1	
	$$\frac{d^3y}{dx^3} = n^3 \cosh^{n-1} x \sinh x - n(n-1)(n-2) \cosh^{n-3} x \sinh x$$ $$\frac{d^4y}{dx^4} = n^3(n-1) \cosh^{n-2} x \sinh^2 x + n^3 \cosh^n x$$ $$-n(n-1)(n-2)(n-3) \cosh^{n-4} x \sinh^2 x - n(n-1)(n-2) \cosh^{n-2} x$$	A1	
		(2)	
	Alternative 1 using $\frac{d^2y}{dx^2} = n^2 y - n(n-1) \cosh^{n-2} x$ leading to $\frac{d^3y}{dx^3} = n^2 \frac{dy}{dx} - \ldots \cosh^{n-3} x \sinh x$ $$\frac{d^4y}{dx^4} = n^2 \frac{d^2y}{dx^2} - \ldots \cosh^{n-4} x \sinh^2 x - \ldots \cosh^{n-2} x$$	M1	

$$\frac{d^3y}{dx^3} = n^2\frac{dy}{dx} - n(n-1)(n-2)\cosh^{n-3}x\sinh x$$ $$\frac{d^4y}{dx^4} = n^2\frac{d^2y}{dx^2} - n(n-1)(n-2)(n-3)\cosh^{n-4}x\sinh^2 x - n(n-1)(n-2)\cosh^{n-2}x$$	A1
	(2)
Alternative 2 $$y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2}x$$ $$y = \cosh^{n-2}x \Rightarrow \frac{d^2y}{dx^2} = \ldots\cosh^{n-2}x - \ldots\cosh^{n-4}x$$ $$\frac{d^4y}{dx^4} = n^2\left[n^2\cosh^n x - n(n-1)\cosh^{n-2}x\right] - n(n-1)\left[\ldots\cosh^{n-2}x - \ldots\cosh^{n-4}x\right]$$	M1
$$y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2}x$$ $$y = \cosh^{n-2}x \Rightarrow \frac{d^2y}{dx^2} = (n-2)^2\cosh^{n-2}x - (n-2)(n-3)\cosh^{n-4}x$$ $$\frac{d^4y}{dx^4} = n^2\left[n^2\cosh^n x - n(n-1)\cosh^{n-2}x\right] - n(n-1)\left[(n-2)^2\cosh^{n-2}x - (n-2)(n-3)\cosh^{n-4}x\right]$$	A1
	(2)

	Alternative 3 Using $\frac{d^2y}{dx^2} = n^2\left(\frac{e^x+e^{-x}}{2}\right)^n - n(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ leading to $\frac{d^3y}{dx^3} =\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right) -\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-3}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^4y}{dx^4} =\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 +\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ $-\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-4}\left(\frac{e^x-e^{-x}}{2}\right)^2 -\ldots\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right) - n(n-1)(n-2)\left(\frac{e^x+e^{-x}}{2}\right)^{n-3}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^4y}{dx^4} = n^3(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + n^3\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$ $-n(n-1)(n-2)(n-3)\left(\frac{e^x+e^{-x}}{2}\right)^{n-4}\left(\frac{e^x-e^{-x}}{2}\right)^2 - n(n-1)(n-2)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}$	A1	1.1b
		(2)	
(b)	When $x = 0$ $y = 1,\quad y' = 0,\quad y'' = n^2 - n(n-1),\quad y^{(3)} = 0,$ $y^{(4)} = n^3 - n(n-1)(n-2)$ Uses their values in the expansion $y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0)+\ldots$	M1	1.1b
	$y = 1 + \frac{nx^2}{2} + \frac{(3n^2-2n)x^4}{24} +\ldots$ cso	A1	2.5
		(2)	
			(8 marks)

4.

5(i)	$\int 2e^{-\frac{1}{2}x}\,dx = -4e^{-\frac{1}{2}x}$	B1	1.1b
	$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx = \lim_{t\to\infty}\left[\left(-4e^{-\frac{1}{2}t}\right)-\left(-4e^{-\frac{1}{2}}\right)\right]$	M1	2.1
	$= 4e^{-\frac{1}{2}}$	A1	1.1b
		(3)	
(ii)(a)	Mean temperature $=\frac{1}{24}\int_0^{24}\left(8-5\sin\left(\frac{\pi}{12}t\right)-\cos\left(\frac{\pi}{6}t\right)\right)dt$	B1	1.2
	$=\frac{1}{24}\left[\left(8t+\frac{60}{\pi}\cos\left(\frac{\pi}{12}t\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right)\right]_0^{24}=\frac{1}{24}[...]$	M1	1.1b
	$=\frac{1}{24}\left[\left(8(24)+\frac{60}{\pi}-\frac{6}{\pi}\times 0\right)-\left(\frac{60}{\pi}\right)\right]=8$ * cso	A1*	2.1
		(3)	
(ii)(b)	E.g. increase the value of the constant 8 / adapt the constant 8 to a function which takes values greater than 8.	B1	3.5c
		(1)	
			(7 marks)

5.

(a)		**DR** $r^2=(-4)^2+\left(\sqrt{48}\right)^2$ or ($r\cos\theta=-4$ and $r\sin\theta=\sqrt{48}$) or $\tan\theta=-\sqrt{3}$ oe	**M1**	2.1	Correct use of relevant formula(e). Some working must be seen.	Correct answer with no working: M0A0
		$r=8$ (ie $z=8e^{i\theta}$) $\theta=2\pi/3$ (ie $z=re^{i2\pi/3}$)	**A1**	1.1	Not ±8 unless later corrected	or eg $\theta=8\pi/3$
		$\sqrt[3]{8}$ or 2	**B1ft**	2.1	Modulus of cube root(s) is the cube root of their modulus	
		$\frac{2\pi}{9}$ soi	**B1ft**	2.1	Argument of (principal) cube root is one third of their argument	
		$\frac{2\pi}{3}+2\pi k$ for $k=1$ and 2 oe seen	**M1**	2.2a	Considering further arguments at angular distance 2π	
		$2e^{\frac{2}{9}\pi i}$, $2e^{\frac{8}{9}\pi i}$ and $2e^{-\frac{4}{9}\pi i}$	**A1**	1.1	or eg $2e^{\frac{2}{9}\pi i}$, $2e^{\frac{8}{9}\pi i}$ and $2e^{\frac{14}{9}\pi i}$	Must be in exponential form, not just $r=$ and $\theta=$. Do not condone any missing i's.
			[6]			
(b)		**DR** The cube roots form an equilateral triangle which has (3) lines of symmetry, (one) through each vertex	**B1**	2.2a		
		$\theta=\frac{2\pi}{9}$, $\theta=\frac{8\pi}{9}$ and $\theta=-\frac{4\pi}{9}$ soi	**B1** **B1**	2.2a 2.2a	for one for all three without extras	ft their angles if $2\pi/3$ apart. If valid alternatives, must come from clear explanation/diagram
			[3]			

6.

(a)	$\dfrac{2x^2+3x+6}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4} \Rightarrow 2x^2+3x+6$ $= A(x^2+4) + (Bx+C)(x+1)$	M1	1.1b
	e.g. $x=-1 \Rightarrow A = \ldots,\ x=0 \Rightarrow C = \ldots,$ coeff $x^2 \Rightarrow B = \ldots$ or Compares coefficients and solves to find values for A, B and C $2 = A+B,\ 3 = B+C,\ 6 = 4A+C$	dM1	1.1b
	$A=1,\quad B=1,\quad C=2$	A1	1.1b
		(3)	
(b)	$\displaystyle\int_0^2 \frac{1}{x+1}+\frac{x+2}{x^2+4}\,\mathrm{d}x = \int_0^2 \frac{1}{x+1}+\frac{x}{x^2+4}+\frac{2}{x^2+4}\,\mathrm{d}x$ $=\left[\alpha\ln(x+1)+\beta\ln(x^2+4)+\lambda\arctan\left(\frac{x}{2}\right)\right]_0^2$	M1	3.1a
	$=\left[ln(x+1)+\frac{1}{2}ln(x^2+4)+arctan\left(\frac{x}{2}\right)\right]_0^2$	A1	2.1
	$=\left[ln(3)+\frac{1}{2}ln(8)+arctan\,1\right]-\left[ln(1)+\frac{1}{2}ln(4)+arctan(0)\right]$ $=$ $=\left[ln(3)+\frac{1}{2}ln(8)+arctan(1)\right]-\left[\frac{1}{2}ln\,4\right]=\underline{ln\left(\frac{3\sqrt{8}}{2}\right)}+\frac{\pi}{4}$	dM1	2.1
	$ln(3\sqrt{2})+\frac{\pi}{4}$	A1	2.2a
		(4)	
			(7 marks)

7.

(a)		RHS $= \cosh^2 x + \sinh^2 x$ $= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2$ $= \frac{1}{4}\left(e^{2x}+e^{-2x}+2+e^{2x}+e^{-2x}-2\right)$ $= \frac{1}{4}\left(2e^{2x}+2e^{-2x}\right) = \frac{1}{2}\left(e^{2x}+e^{-2x}\right)$ $= \cosh 2x =$ LHS	M1 A1 [2]	2.1 1.1	Using definitions of $\cosh x$ **and** $\sinh x$. AG. Intermediate working must be seen.	
(b)		$\cosh^2 x + \sinh^2 x = \cosh 2x$ & $\cosh^2 x - \sinh^2 x = 1$ => $2\cosh^2 x = \cosh 2x + 1$ => $\cosh 2x = 2\cosh^2 x - 1$	B1 [1]	2.2a		
(c)		$10\cosh^2 x - 5 = 16\cosh x + 21$ => $10c^2 - 16c - 26 = 0$ => $5c^2 - 8c - 13 = 0$	M1	1.1	Using the identity from **(b)** to reduce equation to 3 term quadratic	
		$c = -1$ rejected since $\cosh x \geq 1$ (or ≥ 0 oe) or $c = 13/5$	A1	2.3	Both solutions found… … and –1 rejected explicitly with valid reason. E.g "–1 is outside the range of $\cosh x$"	Could be BC
		$\cosh^{-1}\frac{13}{5} = \ln\left(\frac{13}{5}+\sqrt{\left(\frac{13}{5}\right)^2-1}\right) = \ln 5$	M1	1.1	Use of formula for $\cosh^{-1}$ (or by solving quadratic in e^x).	
		$\therefore x = \pm\ln 5$	A1	2.2a	$x = \ln 5$ or $\ln(1/5)$. Must have both solutions.	

8.

$\mathbf{AB} = \begin{pmatrix}10\\12\\10\end{pmatrix}$	*M1		Finding **AB** or an expression for $p_B - p_A$ or $d_B - d_A$	from $\mathbf{r.n} = p_A = 3a - 2a - c$ and $\mathbf{r.n} = p_B = 13a + 10a + 9c$ or $\mathbf{r.\hat{n}} = d_A$ and $\mathbf{r.\hat{n}} = d_B$ ie $22a + 10c$
$\pm 2 = AB\cos\theta = \frac{\lvert\mathbf{AB.n}\rvert}{\lvert\mathbf{n}\rvert}$ with **AB** from above used	dep*M1		Use of dot product to express correct shortest distance in terms of **AB** and the normal to the planes or seeing an appropriate difference between plane constants	or $d_B - d_A = \pm 2$ or $\frac{p_B}{\sqrt{a^2+b^2+c^2}} - \frac{p_A}{\sqrt{a^2+b^2+c^2}} = \pm 2$ allow omission of $\pm$
$\mathbf{n} = \begin{pmatrix}a\\a\\c\end{pmatrix}$ or $\mathbf{n} = \begin{pmatrix}1\\1\\s\end{pmatrix}$ oe	*M1		$\begin{pmatrix}a\\a\\c\end{pmatrix}$ used consistently	
$119a^2 + 110ac + 24c^2 = 0$	A1		Quadratic formed	Or $171a^2 \mp 22a - 24 = 0$ $171c^2 \mp 20c - 119 = 0$ $24s^2 + 110s + 119 = 0$
(7a+4c)(17a+6c)=0 e.g. $a = 4, c = -7 \Rightarrow \mathbf{n}_1 = \begin{pmatrix}4\\4\\-7\end{pmatrix}$	dep*M1		Using one solution of quadratic to obtain a normal of one of the solution planes.	Dependent on all previous M marks
e.g. $a = 6, c = -17 \Rightarrow \mathbf{n}_2 = \begin{pmatrix}6\\6\\-17\end{pmatrix}$	A1		Both correct.	
$\begin{pmatrix}4\\4\\-7\end{pmatrix}.\begin{pmatrix}6\\6\\-17\end{pmatrix} = 167$	Dep*M1		Finding the dot product of their solution normals.	Dependent on all previous M marks
$\cos\theta = \frac{167}{9\times 19} = \frac{167}{171}$ $\Rightarrow$ acute angle is awrt 12.4°	A1 (8)		or awrt 0.217 rads	

9.

Vertices of ABC satisfy $z^3 - 1$ $(= 0)$	B1	1.1	Or $(z-1)\left(z - e^{\frac{2}{3}\pi i}\right)\left(z - e^{\frac{4}{3}\pi i}\right)(= 0)$ Or all three values stated
(Complex number represented by) $M = -\frac{1}{2}$	B1	3.1a	Or one of $M = \frac{1}{2}e^{\pi i}, L = \frac{1}{2}e^{\frac{1}{3}\pi i}, N = \frac{1}{2}e^{-\frac{1}{3}\pi i}$
Vertices of LMN satisfy $8z^3 + 1$ $(= 0)$	M1	3.1a	Attempt at polynomial relating to LMN. $\left(z - \frac{1}{2}e^{\pi i}\right)\left(z - \frac{1}{2}e^{\frac{1}{3}\pi i}\right)\left(z - \frac{1}{2}e^{-\frac{1}{3}\pi i}\right)$ suffices for this mark.
$(z^3 - 1)(8z^3 + 1)$ $(=0)$	M1	2.1	Attempt product of their two cubic factors.
$8z^6 - 7z^3 - 1 = 0$	A1	2.2a	**A0** without justification of $8z^3 + 1 = 0$. Must see $= 0$.
	[5]		
Alternative method			
$A(1),\ B\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right),\ C\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$	B1		Finding A, B, C
$L\left(\frac{1}{4}+\frac{\sqrt{3}}{4}i\right), N\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right),\ M\left(-\frac{1}{2}\right)$	B1		Finding L, M, N
Quadratic satisfying B, C $\quad z^2+z+1=0$ Quadratic satisfying L,N $\quad 4z^2-2z+1=0$ Quadratic satisfying A, M $\quad 2z^2-z-1=0$	M1		Combining to give a 6th degree polynomial
Eqn satisfying all 6 points $(z^2+z+1)(4z^2-2z+1)(2z^2-z-1)=0$	M1		Multiplying out some terms to give quadratics or cubics
$\Rightarrow 8z^6-7z^3-1=0$	A1		Must see = 0
Alternative methods By rotational symmetry Because $\left(e^{\frac{2}{3}\pi i}\right)^3 = 1$ Calculation of vertices of L, M and N. Use of $1 + \omega + \omega^2 = 0$ (and $\omega^3 = 1$) to simplify eg $(z - ½(\omega + 1))(z - ½(\omega^2 + 1)))(z - ½(\omega + \omega^2))$ $= (z + ½\omega^2))(z + ½\omega)))(z + ½)$ etc or to find sum/product of roots.			

10.

9(a)	$\dfrac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+\ldots$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+\ldots$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+\ldots$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{2-(\cos\theta-\mathrm{i}\sin\theta)}{2-(\cos\theta-\mathrm{i}\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\ldots\right\}=\frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\ldots\right\}=\frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\ldots=\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
	Alternative $1+z+z^2+z^3+\ldots$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+\ldots$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+\ldots$	M1	3.1a

	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}e^{i\theta}} \times \frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}} = \frac{4-2e^{-i\theta}}{5-2\left(e^{i\theta}+e^{-i\theta}\right)} = \frac{4-2\left(\cos\theta - i\sin\theta\right)}{5-2\left(2\cos\theta\right)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\ldots=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0 \Rightarrow \cos\theta = 2$	M1	3.1a
	As $(-1\le)\cos\theta\le 1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2\left(5-4\cos\theta\right)}$	M1	3.1a
	$-1\le\cos\theta\le 1$ therefore $\frac{1}{6}<\frac{3}{2\left(5-4\cos\theta\right)}<\frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z}=ki \Rightarrow z = 1+\frac{i}{k}$	M1	3.1a
	mod z >1 contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
			(8 marks)

11.

(a)	Obtains at least two correct solutions	1.1a	M1	$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$
	Obtains all correct solutions	1.1b	A1	$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ (in addition to given solutions)
(b)	Expands $(\cos\theta + \mathrm{i}\sin\theta)^6$	1.1a	M1	By de Moivre's theorem $\cos 6\theta + \mathrm{i}\sin 6\theta = (\cos\theta + \mathrm{i}\sin\theta)^6$ $= \cos^6\theta + 6\cos^5\theta\,(\mathrm{i}\sin\theta) + 15\cos^4\theta\,(\mathrm{i}\sin\theta)^2 + 20\cos^3\theta(\mathrm{i}\sin\theta)^3 + 15\cos^2\theta\,(\mathrm{i}\sin\theta)^4 + 6\cos\theta\,(\mathrm{i}\sin\theta)^5 + (\mathrm{i}\sin\theta)^6$
	Equates real parts	3.1a	M1	
	Obtains correct expression for real part in terms of powers of $\cos\theta$ and $\sin\theta$	1.1b	A1	Equating real parts $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$
	Uses trig identity to express real part in terms of $\cos\theta$	3.1a	M1	$= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$ $= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$
	Completes a rigorous argument to obtain the required result	2.1	R1	$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
(c)	Uses the fact that either $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ is a solution to the first equation to deduce that it is also a solution to the second equation	2.2a	M1	$\theta = \frac{\pi}{4} \Rightarrow \cos 6\theta = 0$ $\Rightarrow 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 0$ from part (b) $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\therefore \left(\cos\theta - \frac{1}{\sqrt{2}}\right)$ is a factor of $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
	Uses the factor theorem	3.1a	M1	Similarly $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\left(\cos\theta + \frac{1}{\sqrt{2}}\right)$ is also a factor of the expression.
	Multiplies their linear factors together	1.1a	M1	So $\left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(\cos\theta + \frac{1}{\sqrt{2}}\right) = \left(\cos^2\theta - \frac{1}{2}\right)$ is a factor and $(2\cos^2\theta - 1)$ is a factor
	Obtains the correct result (oe)	2.1	R1	

(d)	Divides the polynomial by their quadratic factor	3.1a	M1	Let $c = \cos\theta$ $$32c^6 - 48c^4 + 18c^2 - 1 = (2c^2 - 1)(16c^4 - 16c^2 + 1)$$ For $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, $2c^2 - 1 = 0$ So for the other four roots, which are the cosines of the other four solutions to $\cos 6\theta = 0$, $$16c^4 - 16c^2 + 1 = 0$$ Solving this as a quadratic in c^2 gives $$c^2 = \frac{2 \pm \sqrt{3}}{4}$$ So $$c = \pm\sqrt{\frac{2 \pm \sqrt{3}}{4}}$$ Of the angles $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$, and $\frac{11\pi}{12}$, $\frac{11\pi}{12}$ has the negative cosine of the greatest magnitude Therefore $\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\frac{2+\sqrt{3}}{4}}$
	Solves their quartic equation as a quadratic in c^2	1.1a	M1	
	Explains that the roots of the quartic correspond to the cosines of the angles found in part (a)	2.4	E1	
	Obtains all correct roots of the quartic	1.1b	A1	
	Uses a rigorous argument to obtain the required result, including a reason why that particular root corresponds to $\cos\left(\frac{11\pi}{12}\right)$	2.1	R1	