Write yours and your teacher's name at the top of your answer sheets.

U6 Further Mathematics Mock

Paper 1 (Pure)
February 2024
2023-2024

Duration: 2 hours

Total number of marks: 99

Write your answers on file paper.

You are permitted to use a scientific or graphical calculator in this paper.

Final answers should be given to a degree of accuracy appropriate to the context.

Students need a formula booklet.

1.

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}u}(\sinh^{-1}u) = \frac{1}{\sqrt{u^2+1}}$. **[2]**

(b) Find the equation of the normal to the graph of $y = \sinh^{-1} 2x$ at the point where $x = \sqrt{6}$. Give your answer in the form $y = mx + c$ where m and c are given in exact, non-hyperbolic form. **[4]**

2.

(a) Show that $\frac{-3+\sqrt{3}\,\mathrm{i}}{2} = \sqrt{3}\,\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}$. **[2]**

(b) Hence determine the exact roots of the equation $z^5 = \frac{9(-3+\sqrt{3}\,\mathrm{i})}{2}$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[3]**

3.

The line l_1 has equation $\frac{x+5}{1} = \frac{y+4}{-3} = \frac{z-3}{5}$

The plane Π_1 has equation $2x + 3y - 2z = 6$

(a) Find the point of intersection of l_1 and Π_1

(2)

The line l_2 is the reflection of the line l_1 in the plane Π_1

(b) Show that a vector equation for the line l_2 is

$$\mathbf{r} = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

where μ is a scalar parameter.

(5)

The plane Π_2 contains the line l_1 and the line l_2

(c) Determine a vector equation for the line of intersection of Π_1 and Π_2

(2)

The plane Π_3 has equation $\mathbf{r}.\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$ where a and b are constants.

Given that the planes Π_1, Π_2 and Π_3 form a sheaf,

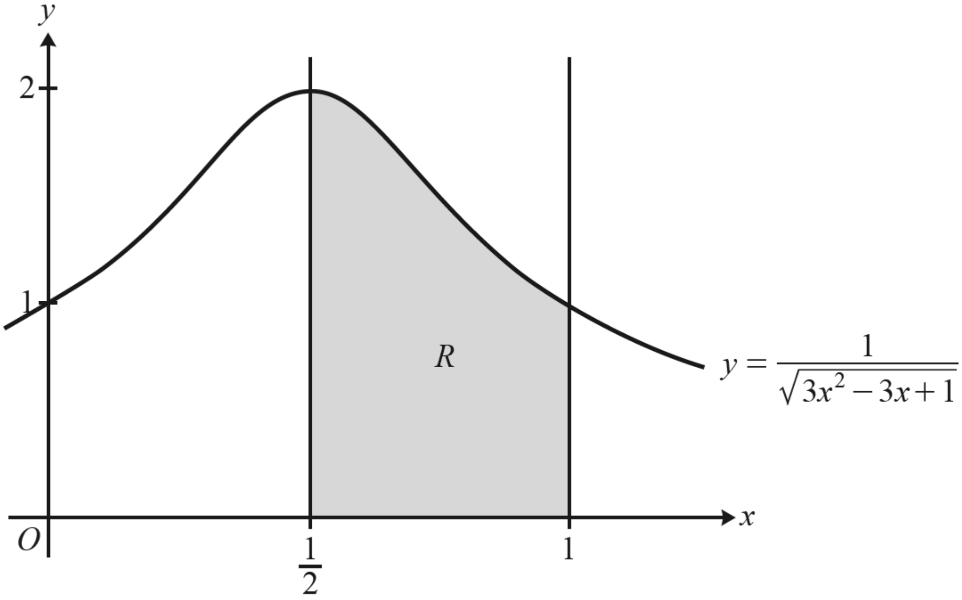
(d) determine the value of a and the value of b.

(3)

4.

In this question you must show detailed reasoning.

The region R is bounded by the curve with equation $y = \dfrac{1}{\sqrt{3x^2 - 3x + 1}}$, the x-axis and the lines with equations $x = \dfrac{1}{2}$ and $x = 1$ (see diagram). The units of the axes are cm.



A pendant is to be made out of a precious metal. The shape of the pendant is modelled as the shape formed when R is rotated by 2π radians about the x-axis.

Find the exact value of the volume of precious metal required to make the pendant, according to the model. **[4]**

5.

A surge in the current, I units, through an electrical component at a time, t seconds, is to be modelled. The surge starts when $t = 0$ and there is initially no current through the component. When the current has surged for 1 second it is measured as being 5 units. While the surge is occurring, I is modelled by the following differential equation.

$$(2t - t^2)\frac{dI}{dt} = (2t - t^2)^{\frac{3}{2}} - 2(t-1)I$$

(a) By using an integrating factor show that, according to the model, while the surge is occurring, I is given by $I = (2t - t^2)(\sin^{-1}(t-1) + 5)$. **[6]**

The surge lasts until there is again no current through the component.

(b) Determine the length of time that the surge lasts according to the model. **[2]**

(c) Determine, according to the model, the rate of increase of the current at the start of the surge. Give your answer in an exact form. **[3]**

6.

An engineer is modelling the motion of a particle P of mass 0.5 kg in a wind tunnel.

P is modelled as travelling in a straight line. The point O is a fixed point within the wind tunnel. The displacement of P from O at time t seconds is x metres, for $t \geqslant 0$.

You are given that $x \geqslant 0$ for all $t \geqslant 0$ and that P does not reach the end of the wind tunnel.

If $t \geqslant 0$, then P is subject to three forces which are modelled in the following way.

- The first force has a magnitude of $5(t+1)\cosh t$ N and acts in the positive x-direction.
- The second force has a magnitude of $0.5x$ N and acts towards O.
- The third force has a magnitude of $\left|\frac{dx}{dt}\right|$ N and acts in the direction of motion of the particle.

(a) The engineer applies the equation "$F = ma$" to the model of the motion of P and derives the following differential equation.

$$5(t+1)\cosh t - 0.5x + \frac{dx}{dt} = 0.5\frac{d^2x}{dt^2}$$

(i) Explain the sign of the $\frac{dx}{dt}$ term in the engineer's differential equation. **[1]**

When $t = 0$ the displacement of P is 6 m, and it is travelling towards O with a speed of $5\,\mathrm{m\,s^{-1}}$.

(ii) Without attempting to solve the differential equation, find the acceleration of P when $t = 0$. **[2]**

Let the particular solution to the differential equation in part **(a)** be a function f such that $x = \mathrm{f}(t)$ for $t \geqslant 0$.

The particular solution to the differential equation can be expressed as a Maclaurin series.

(b) (i) Show that the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t is $6 - 5t$. **[1]**

(ii) Use your answer to part **(a)(ii)** to show that the term in t^2 in the Maclaurin series for $\mathrm{f}(t)$ is $-3t^2$. **[1]**

(iii) By differentiating the differential equation in part **(a)** with respect to t, show that the term in t^3 in the Maclaurin series for $\mathrm{f}(t)$ is $0.5t^3$. **[4]**

You are given that the complete Maclaurin series for the function f is valid for all values of $t \geqslant 0$.

After 0.25 seconds P has travelled 1.43 m towards the origin.

(c) (i) By using the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t^3, evaluate the suitability of the model for determining the displacement of P from O when $t = 0.25$. **[1]**

(ii) Explain why it might not be sensible to use the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t^3 to evaluate the suitability of the model for determining the displacement of P from O when $t = 10$. **[1]**

7.

In this question you must show detailed reasoning.

The power output, p watts, of a machine at time t hours after it is switched on can be modelled by the equation $p = 20 - 20\tanh(1.44t)$ for $t \geqslant 0$.

Determine, according to the model, the **mean** power output of the machine over the first half hour after it is switched on. Give your answer correct to **2** decimal places. **[4]**

8.

In this question you must show detailed reasoning.

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that $\sinh 2x \equiv 2\sinh x \cosh x$. **[2]**

(b) Solve the equation $15\sinh x + 16\cosh x - 6\sinh 2x = 20$, giving all your answers in logarithmic form. **[5]**

9.

The points P, Q and R have coordinates (0, 2, 3), (2, 0, 1) and (1, 3, 0) respectively.

The acute angle between the line segments PQ and PR is θ.

(a) Show that $\sin\theta = \frac{2}{11}\sqrt{22}$. **[3]**

The triangle PQR lies in the plane Π.

(b) Determine an equation for Π, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. **[3]**

The point S has coordinates (5, 3, −1).

(c) By finding the shortest distance between S and the plane Π, show that the volume of the tetrahedron $PQRS$ is $\frac{14}{3}$.

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height] **[4]**

The tetrahedron $PQRS$ is transformed to the tetrahedron $P'Q'R'S'$ by a rotation about the y-axis.

The x-coordinate of S' is $2\sqrt{2}$.

(d) By using the matrix for a rotation by angle θ about the y-axis, as given in the Formulae Booklet, determine in exact form the possible coordinates of R'. **[5]**

10.

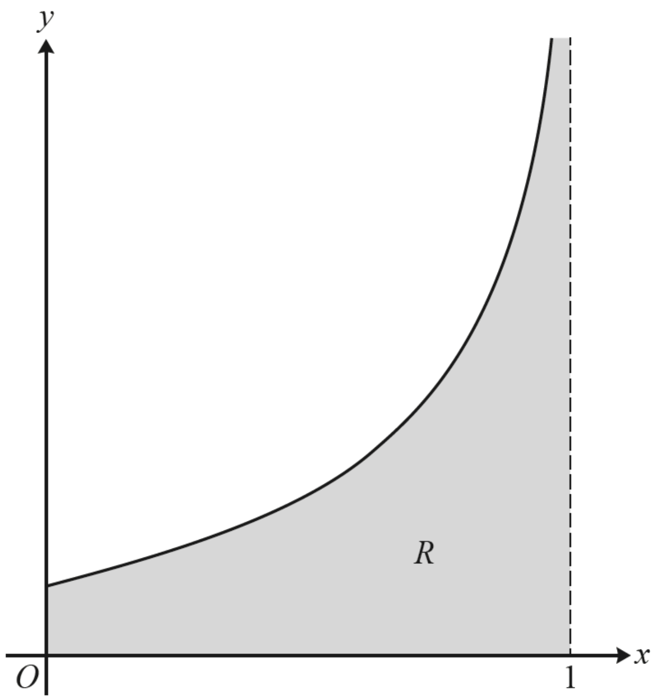
In this question you must show detailed reasoning.

(a) Use de Moivre's theorem to determine constants A, B and C such that $\sin^4\theta \equiv A\cos 4\theta + B\cos 2\theta + C$. **[5]**

The function f is defined by

$$\mathrm{f}(x) = \sin\left(4\sin^{-1}\left(x^{\frac{1}{5}}\right)\right) - 8\sin\left(2\sin^{-1}\left(x^{\frac{1}{5}}\right)\right) + 12\sin^{-1}\left(x^{\frac{1}{5}}\right), \qquad x \in \mathbb{R},\ 0 \leqslant x < 1.$$

(b) Show that $\mathrm{f}'(x) = \dfrac{32}{5\sqrt{1-x^{\frac{2}{5}}}}$. **[6]**



The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{1-x^{\frac{2}{5}}}}$ for $0 \leqslant x < 1$ and the asymptote $x = 1$. The region R is the unbounded region between the curve, the x-axis, the line $x = 0$ and the line $x = 1$.

You are given that the area of R is finite.

(c) Determine the exact area of R. **[3]**

11.

(a) Given that $z = \cos\theta + \text{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n - z^{-n} = 2\text{i}\sin n\theta$$

[2 marks]

(b) The series S is defined as

$$S = \sin\theta + \sin 3\theta + ... + \sin(2n-1)\theta$$

Use part **(a)** to express S in the form

$$S = \frac{1}{2\text{i}}(G_1) - \frac{1}{2\text{i}}(G_2)$$

where each of G_1 and G_2 is a geometric series.

[3 marks]

(c) Hence, show that

$$S = \frac{\sin^2(n\theta)}{\sin\theta}$$

[5 marks]