U6 FM Mock Teacher X 23-24 SOLUTIONS []

1.

(a)		$y=\sinh^{-1}u \Rightarrow \sinh y = u$ $\Rightarrow \cosh y\dfrac{dy}{du}=1 \Rightarrow \dfrac{dy}{du}=\dfrac{1}{\cosh y}$ $\therefore \dfrac{dy}{du}=\dfrac{1}{\pm\sqrt{1+\sinh^2 y}}=\dfrac{1}{\pm\sqrt{u^2+1}}$	M1	2.1	AG. Taking sinh of both sides, differentiating and using $\cosh^2 y-\sinh^2 y=1$	Condone missing ± for M1 Accept $\frac{du}{dy}$ at this stage
		But gradient of $y=\sinh^{-1}u$ is never negative so $\dfrac{dy}{du}=\dfrac{1}{\sqrt{u^2+1}}$	A1	2.4	AG so reason required (accept "always positive")	poor notation can be recovered
		Alternative method: $y=\sinh^{-1}u=\ln\left(u+\sqrt{u^2+1}\right)$ $\therefore \dfrac{dy}{du}=\dfrac{1+\frac{1}{2}\times 2u\left(u^2+1\right)^{-\frac{1}{2}}}{u+\sqrt{u^2+1}}$	M1		AG. Using the definition of $\sinh^{-1}$ in logarithmic form and attempting to differentiate using the chain rule on ln function.	
		$=\dfrac{\left(u^2+1\right)^{\frac{1}{2}}+u}{\left(u^2+1\right)^{\frac{1}{2}}\left(u+\left(u^2+1\right)^{\frac{1}{2}}\right)}=\dfrac{1}{\left(u^2+1\right)^{\frac{1}{2}}}=\dfrac{1}{\sqrt{u^2+1}}$	A1		AG so some intermediate working must be seen. www	
		Alternative method 2: $I=\int\dfrac{1}{\sqrt{u^2+1}}du$ $u=\sinh v \Rightarrow du=\cosh v\,dv$ $\Rightarrow I=\int\dfrac{1}{\sqrt{\sinh^2 v+1}}\cosh v\,dv$ $=\int\dfrac{\cosh v}{\sqrt{\cosh^2 v}}dv=\int 1\,dv=v\ (+c)$ $=\sinh^{-1}u\ (+c)$	M1		Correctly integrates RHS	Condone omission of c
		Differentiating both sides wrt u gives $\dfrac{1}{\sqrt{u^2+1}}=\dfrac{d}{du}\left(\sinh^{-1}u\right)$	A1			
(b)		$y=\sinh^{-1}2x$ $\therefore \dfrac{dy}{dx}=\dfrac{2}{\sqrt{(2x)^2+1}}=\dfrac{2}{\sqrt{4x^2+1}}$	M1	1.1	Differentiating using chain rule or the formula booklet	Giving $\dfrac{1}{\sqrt{\frac{1}{4}+x^2}}$
		$x=\sqrt{6}\Rightarrow y=\sinh^{-1}2\sqrt{6}\ \left(=\ln\left(5+2\sqrt{6}\right)\right)$	M1	1.1	Substituting the x-value into the equation to find the y coordinate of the point	
		$x=\sqrt{6}\Rightarrow \left.\dfrac{dy}{dx}\right\vert_{x=\sqrt{6}}=\dfrac{2}{5}$ $\therefore m=-\dfrac{5}{2}$	M1	1.1	Substituting the x-value into theirgradient and taking negative reciprocal	
		$y-\ln\left(5+2\sqrt{6}\right)=-\dfrac{5}{2}\left(x-\sqrt{6}\right)$ $\therefore y=-\dfrac{5}{2}x+\ln\left(5+2\sqrt{6}\right)+\dfrac{5\sqrt{6}}{2}$	A1	1.1	oe in correct form	
			[4]			

2.

(a)		$r=\dfrac{1}{2}\sqrt{(-3)^2+\left(\sqrt{3}\right)^2}=\dfrac{1}{2}\sqrt{12}=\sqrt{3}$	B1	AG so must show use of $\lvert z\rvert=\sqrt{a^2+b^2}$
		$\arctan\left(-\dfrac{\sqrt{3}}{3}\right)=-\dfrac{\pi}{6}$ $\Rightarrow \theta=-\dfrac{\pi}{6}+\pi=\dfrac{5\pi}{6}$	B1	**AG.** Or $\theta=\pi-\arctan\left(\frac{\sqrt{3}}{3}\right)$, may be indicated on a diagram, but clear reasoning must be shown (eg.finding complementary angle, or use of Pythagoras' theorem and then arcsin or arccos)
		Alternative method: $\sqrt{3}e^{\frac{5}{6}\pi i}=\sqrt{3}\left(\cos\left(\dfrac{5}{6}\pi\right)+i\sin\left(\dfrac{5}{6}\pi\right)\right)$	M1	
		$=\sqrt{3}\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)=\dfrac{\sqrt{3}i-3}{2}$	A1	**AG** Clearly shown
			[2]	
(b)		$r=\left(9\sqrt{3}\right)^{\frac{1}{5}}=\left(3^{\frac{5}{2}}\right)^{\frac{1}{5}}=\sqrt{3}$	B1	For $r=\sqrt{3}$ **oe** (including 1.73....)
		$\theta=\dfrac{1}{5}\left(\dfrac{5}{6}\pi+2r\pi\right)=\dfrac{\pi}{30}(5+12r)$ for $r=0,1,2,3,4$	M1	For their $\frac{5}{6}\pi+2\pi n$ from (a) divided by 5 (either in terms of n, or for at least two values of n).
		$\Rightarrow z=\sqrt{3}e^{\frac{1}{6}\pi i},\ \sqrt{3}e^{\frac{17}{30}\pi i},\sqrt{3}e^{\frac{29}{30}\pi i},\sqrt{3}e^{\frac{41}{30}\pi i},\sqrt{3}e^{\frac{53}{30}\pi i}$	A1	Allow $\sqrt{3}e^{\frac{1}{30}(5+12n)\pi i}$ for $n=0,1,2,3,4$. Accept only $r=\sqrt{3}$ or $(3)^{\frac{1}{2}}$ For last two marks, If M0 then SC B1 for all five roots
			[3]	

3.

(a)	$2(\lambda-5)+3(-3\lambda-4)-2(5\lambda+3)=6 \Rightarrow \lambda=...(-2)$ $\lambda="-2" \Rightarrow x=...$ or $y=...$ or $z=...$ or e.g. $2x+3(-3x-15-4)-2(5x+25+3)=6 \Rightarrow x=...$	**M1**	1.1b
	$(-7, 2, -7)$	**A1**	1.1b
		(2)	
(b)	E.g. $\mathbf{r}=\begin{pmatrix}-5\\-4\\3\end{pmatrix}+\begin{pmatrix}2t\\3t\\-2t\end{pmatrix}$ meets the plane when $2(-5+2t)+3(-4+3t)-2(3-2t)=6 \Rightarrow t=...$	**M1**	3.1a
	$t=2 \Rightarrow$ mirror point is $\begin{pmatrix}-5\\-4\\3\end{pmatrix}+\begin{pmatrix}2\times4\\3\times4\\-2\times4\end{pmatrix}=...$	**M1**	1.1b
	$=\begin{pmatrix}3\\8\\-5\end{pmatrix}$	**A1**	1.1b
	$\mathbf{r}=\begin{pmatrix}-7\\2\\-7\end{pmatrix}+\mu\begin{pmatrix}3-(-7)\\8-2\\-5-(-7)\end{pmatrix}=...$	**ddM1**	1.1b
	$\mathbf{r}=\begin{pmatrix}-7\\2\\-7\end{pmatrix}+\mu\begin{pmatrix}10\\6\\2\end{pmatrix}$ *	**A1***	2.1
		(5)	
	(b) Alternative for first 2 marks:		
	Distance from $(-5, -4, 3)$ to plane is $\frac{\lvert 2\times-5+3\times-4-2\times3-6\rvert}{\sqrt{2^2+3^2+2^2}}=2\sqrt{17}$	**M1**	3.1a
	$\begin{vmatrix}2k\\3k\\-2k\end{vmatrix}=4\sqrt{17} \Rightarrow 4k^2+9k^2+4k^2=16\times17 \Rightarrow k=4$ $k=4 \Rightarrow$ mirror point is $\begin{pmatrix}-5\\-4\\3\end{pmatrix}+\begin{pmatrix}2\times4\\3\times4\\-2\times4\end{pmatrix}=...$	**M1**	1.1b

(c)	Line joining mirror points intersects plane at $\begin{pmatrix}-5\\-4\\3\end{pmatrix}+\begin{pmatrix}2\times2\\3\times2\\-2\times2\end{pmatrix}$, so equation of line is $\mathbf{r}=\begin{pmatrix}-7\\2\\-7\end{pmatrix}+s\begin{pmatrix}-1-(-7)\\2-2\\-1-(-7)\end{pmatrix}=\ldots$	**M1**	3.1a
	$\mathbf{r}=\begin{pmatrix}-7\\2\\-7\end{pmatrix}+s\begin{pmatrix}6\\0\\6\end{pmatrix}$ oe e.g. $\mathbf{r}=\begin{pmatrix}-1\\2\\-1\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}$	**A1**	2.5
		(2)	
	Alternative 1 to (c) (Not on spec)		
	Normal to Π_2: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-3&5\\10&6&2\end{vmatrix}=\begin{pmatrix}-36\\48\\36\end{pmatrix}$ Direction of l_2: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&4&3\\2&3&-2\end{vmatrix}=\begin{pmatrix}-17\\0\\-17\end{pmatrix}$ equation of line is $\mathbf{r}=\begin{pmatrix}-7\\2\\-7\end{pmatrix}+s\begin{pmatrix}-17\\0\\-17\end{pmatrix}$ oe	**M1** **A1**	3.1a 2.5
		(2)	
	Alternative 2 to (c) (Not on spec)		
	Normal to Π_2: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-3&5\\10&6&2\end{vmatrix}=\begin{pmatrix}-36\\48\\36\end{pmatrix}$ $(-3\mathbf{i}+4\mathbf{j}+3\mathbf{k})\cdot(-7\mathbf{i}+2\mathbf{j}-7\mathbf{k})=8$ Π_2 is $3x-4y-3z=-8$ then e.g. solves simultaneously with Π_1 and $x=\lambda$ to give $y=2$, $z=\lambda$ So equation of line is $\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}$ oe	**M1** **A1**	3.1a 2.5
		(2)	
	Alternative 3 to (c) (Not on spec)		
	As alternative 2 to find the equation of plane 2: $3x-4y-3z=-8$ Then solves simultaneously with plane 1 to give e.g. $y=2$, $x=z$ Hence $\mathbf{r}=\begin{pmatrix}s\\2\\s\end{pmatrix}$ oe	**M1** **A1**	3.1a 2.5
		(2)	

(d)	Line from (c) must lie in plane, so $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is perpendicular to $\begin{pmatrix}1\\1\\a\end{pmatrix}$ $\Rightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\a\end{pmatrix}=0\Rightarrow 1\times1+0\times1+1\times a=0\Rightarrow a=...$	**M1**	3.1a
	$a=-1$	**A1**	1.1b
	$b=\begin{pmatrix}-7\\2\\-7\end{pmatrix}.\begin{pmatrix}1\\1\\-1\end{pmatrix}=2$	**A1**	2.2a
		(3)	
	Alternative 1 to (d):		
	$\begin{pmatrix}-7\\2\\-7\end{pmatrix}.\begin{pmatrix}1\\1\\a\end{pmatrix}=b\Rightarrow -7+2-7a=b$ $\Rightarrow a=...$ or $b=...$ $\begin{pmatrix}-1\\2\\-1\end{pmatrix}.\begin{pmatrix}1\\1\\a\end{pmatrix}=b\Rightarrow -1+2-a=b$	**M1**	3.1a
	$a=-1$ or $b=2$	**A1**	1.1b
	$a=-1$ and $b=2$	**A1**	2.2a
		(2)	
	Alternative 2 to (d) (Not on spec):		
	Normal to Π_2: $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & -3 & 5\\10 & 6 & 2\end{vmatrix}=\begin{pmatrix}-36\\48\\36\end{pmatrix}$ $\begin{vmatrix}3 & -4 & -3\\2 & 3 & -2\\1 & 1 & a\end{vmatrix}=0\Rightarrow 3(3a+2)+4(2a+2)-3(-1)=0\Rightarrow a=...$	**M1**	3.1a
	$a=-1$	**A1**	1.1b
	$a=-1$ and $b=2$	**A1**	2.2a
		(2)	
	Alternative 3 to (d):		
	$\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}$ lies in $\Pi_3\Rightarrow s+2+as=b$ $(a+1)s+2=b\Rightarrow a=...$	**M1**	3.1a
	$a=-1$	**A1**	1.1b
	$a=-1$ and $b=2$	**A1**	2.2a
		(2)	

4.

DR $V=\pi\int_{\frac{1}{2}}^{1}\left((3x^2-3x+1)^{-\frac{1}{2}}\right)^2 \mathrm{d}x$	**M1**	3.3	Using $V=\pi\int_a^b y^2\mathrm{d}x$ with limits	Accept squared out expression Condone omission of dx
$V=\pi\int\frac{1}{3x^2-3x+1}\mathrm{d}x=\frac{1}{3}\pi\int\frac{1}{x^2-x+\frac{1}{3}}\mathrm{d}x$ $=\frac{1}{3}\pi\int\frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{3}}\mathrm{d}x$ $=\frac{1}{3}\pi\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{12}}\mathrm{d}x$	**M1**	2.2a	Expressing the integral in completed square form	or $\pi\int\frac{1}{3\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}\mathrm{d}x$ or $\pi\int\frac{1}{\left(\sqrt{3}x-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}\mathrm{d}x$
$\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{12}}\mathrm{d}x=\left[\frac{1}{\sqrt{\frac{1}{12}}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}\right)\right]$	**A1**	1.1	$=\left[\frac{2}{\sqrt{3}}\pi\tan^{-1}\left(\sqrt{3}(2x-1)\right)\right]$ may be equivalent based on their form and their substitution	Could see a substitution eg $u=\sqrt{3}\left(x-\frac{1}{2}\right)$ with $\mathrm{d}u=\sqrt{3}\mathrm{d}x$
$=\frac{2}{\sqrt{3}}\pi\left[\tan^{-1}\left(\sqrt{3}\right)-\tan^{-1}(0)\right]$ $=\frac{2}{\sqrt{3}}\pi\frac{\pi}{3}=\frac{2\sqrt{3}}{9}\pi^2$ so $\frac{2\sqrt{3}}{9}\pi^2$ cm^3	**A1**	3.4	oe	Do not penalise missing units
	[4]			

5.

(a)		$\left(2t-t^2\right)\frac{dI}{dt}=\left(2t-t^2\right)^{\frac{3}{2}}-2(t-1)I$ $\therefore \frac{dI}{dt}=\left(2t-t^2\right)^{\frac{1}{2}}-\frac{2(t-1)}{2t-t^2}I$ $\therefore \frac{dI}{dt}+\frac{2(t-1)}{2t-t^2}I=\left(2t-t^2\right)^{\frac{1}{2}}$	***M1**	3.3	Rearranging to the form $\frac{dI}{dt}+P(t)I=Q(t)$	
		$\text{IF}=e^{\int\frac{2(t-1)}{2t-t^2}dt}=e^{-\int\frac{2-2t}{2t-t^2}dt}=e^{-\ln(2t-t^2)}=\frac{1}{2t-t^2}$	**A1**	2.2a	Finding correct integrating factoras an expression not involving exponentials and logs.	Ignore unnecessary constant multiplier here
		$\therefore\left(2t-t^2\right)^{-1}\frac{dI}{dt}+2(t-1)\left(2t-t^2\right)^{-2}I=\left(2t-t^2\right)^{-\frac{1}{2}}$ $\frac{d}{dt}\left(\left(2t-t^2\right)^{-1}I\right)=\left(2t-t^2\right)^{-\frac{1}{2}}$	***dep*M1**	1.1	Multiplying both sides by IF and recognising LHS as exact derivative of $\left(2t-t^2\right)^{-1}I$	Can be implied by next **M1** Can be awarded from slip in initial rearrangement if their IF works for their rearrangement
		$\left(2t-t^2\right)^{-1}I=\int\left(2t-t^2\right)^{-\frac{1}{2}}dt=\int\frac{1}{\sqrt{1-(t-1)^2}}dt$	**dep*M1**	1.1	Taking integral of both sides and attempt to complete square in order to express RHS in a standard form (or using the substitution $u = t - 1$ including $du = dt$).	Condone $\pm1\pm(t-1)^2$ for **M1** $=\int\frac{1}{\sqrt{1-u^2}}du$
		$=\sin^{-1}(t-1)+c$	***A1**	1.1	For correctly integrating RHS to a function of t. "+ c" not necessary here.	
		$t=1,\ I=5\Rightarrow(2-1)^{-1}5=\sin^{-1}(1-1)+c$ $\therefore c=5$ $\therefore\left(2t-t^2\right)^{-1}I=\sin^{-1}(t-1)+5$ $\therefore I=(2t-t^2)(\sin^{-1}(t-1)+5)$	**dep*A1**	3.3	AG so use of relevant condition must be explicit. Verification of AG by substitution is not sufficient; value of c must be derived.	Ignore workings using other condition ($t = 0$, $I = 0$)
			[6]			
(b)		$I=0\Rightarrow(2t-t^2)(\sin^{-1}(t-1)+5)=0$ $\therefore t(2-t)(\sin^{-1}(t-1)+5)=0$ $\therefore t=2$ So length of surge is 2 – 0 = 2 (seconds)	**B1**	3.1a	If no other later comment or statement to the contrary then accept just $t=2$	
		or $\sin^{-1}(t-1)+5=0$ $\therefore\sin^{-1}(t-1)=-5$ which is not possible (since $-\frac{1}{2}\pi\le\sin^{-1}(t-1)\le\frac{1}{2}\pi$.)	**B1**	2.4	Do not accept incorrect explanation eg $-1\le\sin^{-1}(t-1)\le1$	If **B0B0** then**Sc1**if length of surge determined to be awrt 1.96 s from sin(–5rads) + 1 after $t = 2$ found
			[2]			
(c)		$\left(I=(2t-t^2)(\sin^{-1}(t-1)+5)\right)\pm1\pm(t-1)^2$ & $\left(2t-t^2\right)\frac{dI}{dt}=\left(2t-t^2\right)^{\frac{3}{2}}-2(t-1)I$ $\therefore\frac{dI}{dt}=\frac{(2t-t^2)^{\frac{3}{2}}-2(t-1)(2t-t^2)(\sin^{-1}(t-1)+5)}{2t-t^2}$	***M1**	3.4	Finding an expression for $\frac{dI}{dt}$ in terms of t by either eliminating given I from given DE or differentiating given $I(t)$using product rule.	$\left(I=(2t-t^2)(\sin^{-1}(t-1)+5)\Rightarrow\right)\frac{dI}{dt}=$ $2(1-t)(\sin^{-1}(t-1)+5)+\frac{(2t-t^2)}{\sqrt{1-(t-1)^2}}$
		$\therefore\frac{dI}{dt}=\left(2t-t^2\right)^{\frac{1}{2}}-2(t-1)(\sin^{-1}(t-1)+5)$	**M1 dep ***	2.2a	Simplifying their $\frac{dI}{dt}$ so that there is no denominator which is zero at $t = 0$	$\therefore\frac{dI}{dt}=$ $2(1-t)(\sin^{-1}(t-1)+5)+\frac{(2t-t^2)}{\sqrt{2t-t^2}}$ $=(2-2t)(\sin^{-1}(t-1)+5)+(2t-t^2)^{\frac{1}{2}}$
		$\therefore t=0$ $\Rightarrow\frac{dI}{dt}=0-2(-1)(\sin^{-1}(-1)+5)$ $=2\left(5-\frac{1}{2}\pi\right)=10-\pi$ (units/s)	**A1**	3.4	Must be in a simplified non-trigonometric form. Ignore units.	If M1M0 or M0M0, then **Sc B1** for correct answer
			[3]			

6.

(a)	(i)	e.g. sign of $\frac{dy}{dt}$ is +ve because the force is in the same direction of motion	B1	Convincingly shown
			[1]	
	(ii)	$\frac{1}{2}\frac{d^2x}{dt^2} = 5\cosh 0 - 3 - 5$ $= -3$ $\Rightarrow \frac{d^2x}{dt^2} = -6(\text{ms}^{-2})$	M1 A1	Substitutes $t = 0$ and $\frac{dx}{dt} = \pm 5$ Or 6 towards O
			[2]	
(b)	(i)	Maclaurin: $x = f(0) + f'(0)t + \ldots$ When $t = 0, (x = 6 =) f(0) = 6, \left(\frac{dx}{dt} =\right) = f'(0) = -5$ $\Rightarrow x = 6 - 5t\ldots.$	B1	First two terms ofMaclaurin's formula, possibly in generalised form, must be seen **AG** Convincingly shown.
			[1]	
	(ii)	3rd term of Maclaurin is $f''(0)\frac{t^2}{2!}$ With (from (a)(ii)) $f''(0) = -6$ So 3rd term is $(-6)\frac{t^2}{2!} = -3t^2$	B1	**AG** Convincingly shown
			[1]	
(b)	(iii)	$\Rightarrow 5\cosh t + 5(t+1)\sinh t - 0.5\frac{dx}{dt} + \frac{d^2x}{dt^2} = \frac{1}{2}\frac{d^3x}{dt^3}$ When $t = 0,\quad 5 + 0 + 2.5 - 6 = \frac{1}{2}f'''(0)$ $\Rightarrow f'''(0) = 3$ $\Rightarrow$ 4th term $= f'''(0)\frac{t^3}{3!} = \frac{1}{2}t^3$	M1 A1 M1 A1	For $\frac{d}{dt}(\cosh t) = \sinh t$ and product rule attempted Fully correct differentiation Substitute $t = 0$ **AG**. Allow embedded answer.
			[4]	
(c)	(i)	$x = 6 - 5t - 3t^2 + \frac{1}{2}t^3$ $\Rightarrow$ When $t = 0.25,\ x = 6 - 1.25 - 0.1875 - 0.0078 \approx 4.570\ldots$ $\Rightarrow$ Distance travelled $6 - 4.570 \approx 1.430\ldots$ So suitable as value close	B1	Substitutes $t = 0.25$ to obtain an approximation and correct conclusion
			[1]	
	(ii)	e.g. more terms may be required Higher terms may be large The candidate calculates the term in t^4 ($11t^4/12$) and indicates that this term is large for values of $t > 1$.	B1	Allow any correct explanation that explains/implies that for $t > 1$ some of the higher power terms are large and so non-negligible. f(10) = 156 is too large is **not** enough
			[1]	

7.

DR Mean value $= \frac{1}{0.5}\int_0^{0.5}(20 - 20\tanh(1.44t))\,dt$ $= \left[40t - \frac{40}{1.44}\ln\left(\left\lvert e^{1.44t} + e^{-1.44t}\right\rvert\right)\right]_0^{0.5}$ $= \left(20 - \frac{40}{1.44}\left(\ln\left(e^{0.72} + e^{-0.72}\right) - \ln 2\right)\right)$ $= \left(20 - \frac{40}{1.44}\ln\frac{2.5412}{2}\right) = 13.35(\text{W})$	B1 M1 A1 A1	For using the definition of the mean value of p wrt t, correct limits and 1/0.5. For $\int \tanh 1.44t\ dt = k\ln\lvert e^{1.44t} + e^{-1.44t}\rvert(+c)$ k can = 1 Or $\int \tanh 1.44t\ dt = k\ln\lvert\cosh 1.44t\rvert(+c)$ May see integration by substitution, eg.$u = e^{1.44t} + e^{-1.44t}$or $u = \cosh 1.44t$. If so, award this mark for $k\ln\lvert u\rvert$ seen For fully correct integration of tanh $1.44t$. Either for $\int \tanh 1.44t\ dt = \frac{1}{1.44}\ln\lvert e^{1.44t} + e^{-1.44t}\rvert(+c)$ or $\int \tanh 1.44t\ dt = \frac{1}{1.44}\ln\lvert\cosh 1.44t\rvert(+c)$ or $\int \tanh 1.44t\ dt = \frac{1}{1.44}\ln\lvert u\rvert\,(+c)$ with u as above. Condone missing modulus. cao, with clear working.
	[4]	

8.

(a)		**DR** $(\text{RHS} = 2\sinh x\cosh x =) 2\times\frac{e^x - e^{-x}}{2}\times\frac{e^x+e^{-x}}{2}$	**M1**	2.1	AG. Use of exponential definition of sinh or cosh	Must be **used** on LHS or RHS
		$=\frac{1}{2}\left(e^{2x}+e^0-e^0-e^{-2x}\right)$ $=\frac{e^{2x}-e^{-2x}}{2}=\sinh 2x=\text{LHS}$	**A1**	2.2a	AG. Soan intermediate step must be shown LHS must be equated to RHS	e.g. $2\left(\frac{e^{2x}-e^{-2x}}{4}\right)$ Accept full reverse argument
			[2]			
(b)		**DR** 15sinh$x$ + 16 cosh$x$ – 12sinh$x$cosh$x$ = 20	**B1**	3.1a	Use of identity in **(a)**.	
		$12sc - 16c - 15s + 20 = 0$ $(3s - 4)(4c - 5) = 0$	**M1**	1.1	Writing as = 0 and factorising	(wheres $= \sinh x$ and $c = \cosh x$)
		$x=\sinh^{-1}\left(\frac{4}{3}\right)$ or $x=\cosh^{-1}\left(\frac{5}{4}\right)$	**A1**	1.1	Complete solution in any form (assume that $\cosh^{-1}$ is multi-valued here)	
		$\sinh^{-1}\frac{4}{3}=\ln\left(\frac{4}{3}+\sqrt{\left(\frac{4}{3}\right)^2+1}\right)=\ln 3$	**A1**	1.1		
		$\cosh^{-1}\frac{5}{4}=\pm\ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1}\right)=\pm\ln 2$	**A1**	3.2a	Must show ± explicitly (or have both ln2 and ln½)	
		Alternative method: $15\frac{e^x-e^{-x}}{2}+16\frac{e^x+e^{-x}}{2}-6\frac{e^{2x}-e^{-2x}}{2}=20$	**B1**		Use of exponential definitions of sinh$x$, cosh$x$ and sinh2x in equation	Also award if starts with main method before using exponentials
		$\therefore 15e^x-15e^{-x}+16e^x+16e^{-x}-6e^{2x}+6e^{-2x}=40$ $\therefore 15e^{3x}-15e^x+16e^{3x}+16e^x-6e^{4x}+6=40e^{2x}$ $\therefore 6e^{4x}-31e^{3x}+40e^{2x}-e^x-6=0$	***M1**		Multiplying by e^{2x} and collecting like terms to write as quartic equation in e^x	Could see a substitution eg $y = e^x$ leading to $6y^4-31y^3+40y^2-y-6=0$ Could use Pythagoras to derive quartic in sinh or cosh
		$y=e^x \Rightarrow 6y^4-31y^3+40y^2-y-6=0$ $6\times16-31\times8+40\times4-2-6=256-256=0$ $6y^3(y-2)-19y^2(y-2)+2y(y-2)+3(y-2)=0$ $(y-2)(6y^3-19y^2+2y+3)=0$	***dep* M1**		Using factor theorem to deduce that $e^x = 2$ (or 3 or ½) is a solution and factorising.	or $(6y^2-y-1)(y^2-5y+6)$ seen
		$6\times27-19\times9+2\times3+3=171-171=0$ $6y^3-19y^2+2y+3=$ $6y^2(y-3)-y(y-3)-(y-3)$ $=(y-3)(6y^2-y-1)=(y-3)(2y-1)(3y+1)$	**dep*M 1**		Using factor theorem to find another factor and fully factorising	
		$\therefore y = e^x = 2$, ½, 3 or –⅓. But $e^x > 0$ $\therefore x =$ ln2, ln½ (or –ln2) or ln3 only	**A1**		Must reject negative root explicitly for **A1**	**ScB1** for correct solution after B1M1M0M0
			[5]			

9.

(a)		Alternative method: $\lvert QR\rvert=\sqrt{11},\ \lvert RP\rvert=\sqrt{11},\ \lvert PQ\rvert=\sqrt{12}$	**B1**	Sides of triangle
		Cosine rule $\Rightarrow \cos\theta=\dfrac{12+11-11}{2\times\sqrt{11}\times\sqrt{12}}=\sqrt{\dfrac{3}{11}}$	**M1**	Cosine rule
		$\Rightarrow \sin\theta=\sqrt{1-\dfrac{3}{11}}=\sqrt{\dfrac{8}{11}}=\dfrac{2}{11}\sqrt{22}$	**A1**	
		Or: $\lvert QR\rvert=\sqrt{11},\ \lvert RP\rvert=\sqrt{11},\ \lvert PQ\rvert=\sqrt{12}$	**B1**	Sides of triangle
		Drop perpendicular from R to QP at M	**M1**	Recognises an isosceles triangle so uses a median line
		$RM=\sqrt{11-\left(\dfrac{1}{2}\sqrt{12}\right)^2}=\sqrt{8}$	**A1**	
		$\Rightarrow \sin\theta=\sqrt{\dfrac{8}{11}}=\dfrac{2}{11}\sqrt{22}$		
			[3]	
(a)		$\overrightarrow{PQ}=\begin{pmatrix}2\\-2\\-2\end{pmatrix},\ \overrightarrow{PR}=\begin{pmatrix}1\\1\\-3\end{pmatrix}$	**B1**	Both correct or other way round
		$\cos\theta=\dfrac{\begin{pmatrix}2\\-2\\-2\end{pmatrix}.\begin{pmatrix}1\\1\\-3\end{pmatrix}}{\sqrt{(2)^2+(-2)^2+(-2)^2}\times\sqrt{(1)^2+(1)^2+(-3)^2}}$ $=\dfrac{6}{\sqrt{12}\sqrt{11}}$	**M1**	Correct use of scalar product including correct method for magnitudes and dot product
		$\Rightarrow \sin\theta=\left(\sqrt{1-\left(\dfrac{6}{\sqrt{12}\sqrt{11}}\right)^2}\right)=\sqrt{\dfrac{8}{11}}=2\sqrt{\dfrac{22}{11^2}}=\dfrac{2}{11}\sqrt{22}$	**A1**	**AG** All working must be using exact forms
		Alternative for M1 A1 $\sin\theta=\dfrac{\left\lvert\begin{pmatrix}2\\-2\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\-3\end{pmatrix}\right\rvert}{\sqrt{(2)^2+(-2)^2+(-2)^2}\times\sqrt{(1)^2+(1)^2+(-3)^2}}$	**M1**	Correct use of vector product to find $\lvert\sin\theta\rvert$ including correct method for magnitudes, and correct method for $\begin{pmatrix}2\\-2\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\-3\end{pmatrix}$ soi. Condone $\sin\theta$ instead of $\lvert\sin\theta\rvert$
		$=\dfrac{4\left\lvert\begin{pmatrix}2\\1\\1\end{pmatrix}\right\rvert}{\sqrt{12}\sqrt{11}}=\dfrac{4\sqrt{6}}{\sqrt{132}}=\dfrac{4}{\sqrt{22}}=\dfrac{2}{11}\sqrt{22}$	**A1**	**AG**.
			[3]	
(b)		$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{pmatrix}2\\-2\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\-3\end{pmatrix}=4\begin{pmatrix}2\\1\\1\end{pmatrix}$	**B1**	BC or any other relevant vector product. Can be awarded if seen in (a)
		$\Rightarrow 2x+y+z=d$	**M1**	Attempt to use **r.n** = $\boldsymbol{d}$ to form linear equation. **ft** *their* vector product.
		Sub for a point $\Rightarrow d=5 \quad \Rightarrow 2x+y+z=5$	**A1**	Substitutes coordinates of P, Q or R to find d. (multiples accepted)
			[3]	
(c)		$D=\dfrac{\left\lvert\begin{pmatrix}5\\3\\-1\end{pmatrix}.\begin{pmatrix}2\\1\\1\end{pmatrix}-5\right\rvert}{\left\lvert\begin{pmatrix}2\\1\\1\end{pmatrix}\right\rvert}=\dfrac{7}{\sqrt{6}}$	**M1** **A1**	Uses the formula given in formula book, or any other complete method for the shortest distance, **ft** *their* $\overrightarrow{PQ}\times\overrightarrow{PR}$. Correct shortest distance.
		$V=\dfrac{1}{3}\times\left(\dfrac{1}{2}\lvert\overrightarrow{PQ}\rvert\lvert\overrightarrow{PR}\rvert\sin\theta\right)\times D$ $=\dfrac{1}{3}\times\dfrac{1}{2}\times2\sqrt{3}\times\sqrt{11}\times\dfrac{2}{11}\sqrt{22}\times\dfrac{7}{\sqrt{6}}$	**M1**	Uses $\frac{1}{2}\lvert\overrightarrow{PQ}\rvert\lvert\overrightarrow{PR}\rvert\sin\theta$oe, multiplied by their D/3, or $\frac{1}{2}\left\lvert\begin{pmatrix}2\\-2\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\-3\end{pmatrix}\right\rvert$ multiplied by their D/3, but must indicate that $\frac{1}{2}\left\lvert\begin{pmatrix}2\\-2\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\-3\end{pmatrix}\right\rvert$ is the area of the base.
		$=\dfrac{14}{3}$	**A1**	**AG**.
			[4]	

(d)		$\begin{pmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{pmatrix}\begin{pmatrix}5\\3\\-1\end{pmatrix}=\begin{pmatrix}2\sqrt{2}\\ \ldots\\ \ldots\end{pmatrix}$ $\Rightarrow 5\cos\phi-\sin\phi=2\sqrt{2}$	**M1**	For rotation matrix multiplied by $\overrightarrow{OR}$ or $\overrightarrow{OS}$.
		$\Rightarrow \sin^2\phi=\left(5\cos\phi-2\sqrt{2}\right)^2$ $\Rightarrow 1-\cos^2\phi=25\cos^2\phi-20\sqrt{2}\cos\phi+8$ $\Rightarrow 26\cos^2\phi-20\sqrt{2}\cos\phi+7=0$	**M1**	For a correct step to form quadratic equation in $\sin\phi$ or $\cos\phi$ only. For reference: $26\sin^2\phi+4\sqrt{2}\sin\phi-17=0$
		$\Rightarrow \cos\phi=\frac{\sqrt{2}}{2},\ \frac{7\sqrt{2}}{26}$	**A1**	Solves quadratic equation in $\sin\phi$ or $\cos\phi$. (Exact answers required)
		$\sin\phi=5\cos\phi-2\sqrt{2}=\frac{\sqrt{2}}{2},\ -\frac{17\sqrt{2}}{26}$ $\begin{pmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{pmatrix}\begin{pmatrix}1\\3\\0\end{pmatrix}=\begin{pmatrix}\cos\phi\\3\\-\sin\phi\end{pmatrix}$	**M1**	Uses their $\sin\phi$ or $\cos\phi$ with $5\cos\phi-\sin\phi=2\sqrt{2}$to find $\cos\phi$ or $\sin\phi$ respectively, (even if only one root) or from$\sin\phi=\sqrt{1-\cos^2\phi}$ or $\cos\phi=\sqrt{1-\sin^2\phi}$ (condone inclusion of $\pm$), or repeats previous method and multiplies out the matrices.
		$\Rightarrow \begin{pmatrix}\frac{\sqrt{2}}{2}\\3\\-\frac{\sqrt{2}}{2}\end{pmatrix},\ \begin{pmatrix}\frac{7\sqrt{2}}{26}\\3\\\frac{17\sqrt{2}}{26}\end{pmatrix}$ i.e. $R'=\left(\frac{\sqrt{2}}{2},3,-\frac{\sqrt{2}}{2}\right)$ or $\left(\frac{7\sqrt{2}}{26},3,\frac{17\sqrt{2}}{26}\right)$	**A1**	For both, and no others. Accept given as $\overrightarrow{OR'}$. If M0M0A0M0A0, SCB1 for any $R'$with y-coordinate = 3, and no other y-coordinates.
			[5]	

(d)		**Alternative method for first 3 marks:** $5\cos\phi-\sin\phi=2\sqrt{2}$	**M1**	For rotation matrix multiplied by $\overrightarrow{OR}$ or $\overrightarrow{OS}$.
		$\Rightarrow 5\cos\phi-\sin\phi=\sqrt{26}\cos(\phi+\alpha)=2\sqrt{2}$ where $\tan\alpha=\frac{1}{5}\Rightarrow \sin\alpha=\frac{1}{\sqrt{26}},\cos\alpha=\frac{5}{\sqrt{26}}$ $\sqrt{26}\sin(\phi+\alpha)=\pm\sqrt{26-\left(2\sqrt{2}\right)^2}=\pm 3\sqrt{2}$	**M1**	Expressing in the form $R\cos(\phi+\alpha)$ or $R\sin(\phi+\alpha)$ oe. Note that $5\cos\phi-\sin\phi$ $=\sqrt{26}\sin\left(\phi+\arctan\left(-\frac{1}{5}\right)+\pi\right)$
		$\cos\phi=\cos\left((\phi+\alpha)-\alpha\right)=\cos(\phi+\alpha)\cos\alpha+\sin(\phi+\alpha)\sin\alpha$ $=\frac{2\sqrt{2}}{\sqrt{26}}\times\frac{5}{\sqrt{26}}\pm\frac{3\sqrt{2}}{\sqrt{26}}\times\frac{1}{\sqrt{26}}$ $\Rightarrow \cos\phi=\frac{\sqrt{2}}{2},\ \frac{7\sqrt{2}}{26}$	**A1**	Solves for $\sin\phi$ or $\cos\phi$. For reference,$\frac{\sqrt{2}}{2}\approx 0.707,-\frac{17\sqrt{2}}{26}\approx -0.925$ and $\frac{7\sqrt{2}}{26}\approx 0.381$.
			[3]	

10.

(a)		**DR** $e^{i\theta}-e^{-i\theta}=2i\sin\theta$ $\Rightarrow\left(e^{i\theta}-e^{-i\theta}\right)^4=16\sin^4\theta$ $\Rightarrow\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{-4i\theta}\right)$ $\Rightarrow\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{-4i\theta}\right)=\left(e^{4i\theta}+e^{-4i\theta}\right)-4\left(e^{2i\theta}+e^{-2i\theta}\right)+6$ $\Rightarrow 2\cos4\theta-8\cos2\theta+6=16\sin^4\theta$ $\Rightarrow\sin^4\theta=\frac{1}{8}\cos4\theta-\frac{1}{2}\cos2\theta+\frac{3}{8}$ i.e. $A=\frac{1}{8},B=-\frac{1}{2},C=\frac{3}{8}$	**B1** **M1** **M1** **M1** **A1**	Or $e^{i\theta}+e^{-i\theta}=2\cos\theta$ May use z without definition **oe**, eg. $(2i\sin\theta)^4=16\sin^4\theta=\left(e^{i\theta}-e^{-i\theta}\right)^4$. Award this mark for $\sin\theta$ to the power of four, and for $(2i)^4=16$. Note that 16 may appear later. Expanding $(e^{i\theta}-e^{-i\theta})^4$ with correct coefficients. Grouping terms and using $e^{i\theta}+e^{-i\theta}=2\cos\theta$. cao, from fully correct reasoning. Allow A, B, C seen in the expression only.
			[5]	
(b)		**DR** Let $u=x^{\frac{1}{5}}\Rightarrow\frac{du}{dx}=\frac{1}{5}x^{-\frac{4}{5}}$ Let $v=\sin^{-1}u\Rightarrow\frac{dv}{du}=\frac{1}{\sqrt{1-u^2}}=\frac{1}{\sqrt{1-x^{\left(\frac{2}{5}\right)}}}$ $\Rightarrow f=\sin4v-8\sin2v+12v\Rightarrow\frac{df}{dv}=4\cos4v-16\cos2v+12$ $\frac{df}{dx}=\frac{df}{dv}.\frac{dv}{du}.\frac{du}{dx}=\left(4\cos4v-16\cos2v+12\right)\frac{1}{\sqrt{1-x^{\left(\frac{2}{5}\right)}}}\times\frac{1}{5}x^{-\frac{4}{5}}$ $\frac{df}{dv}=32\left(\frac{1}{8}\cos4v-\frac{1}{2}\cos2v+\frac{3}{8}\right)=32(\sin v)^4=32u^4=32x^{\frac{4}{5}}$ Then $\frac{df}{dx}=\frac{df}{dv}.\frac{dv}{du}.\frac{du}{dx}=32x^{\frac{4}{5}}\times\frac{1}{\sqrt{1-x^{\frac{2}{5}}}}\times\frac{1}{5}x^{-\frac{4}{5}}=\frac{32}{5\sqrt{1-x^{\frac{2}{5}}}}$	**B1** **M1** **A1** **M1** **M1** **A1**	Sight of $\frac{d\left(\sin^{-1}u\right)}{du}=\frac{1}{\sqrt{1-u^2}}$ Uses chain rule Correct derivative, f' Uses result from (a) Uses $\sin^4\left(\sin^{-1}\left(x^{\frac{1}{5}}\right)\right)=x^{\frac{4}{5}}$ **AG** Clearly shown
			[6]	
(c)		$R=\lim_{k\to1}\int_0^k\frac{1}{\sqrt{1-x^{\frac{2}{5}}}}dx=\frac{5}{32}\lim_{k\to1}[f(x)]_0^k=\frac{5}{32}\lim_{k\to1}(f(k)-f(0))$ $f(0)=0$ $f(k)=\sin\left(4\sin^{-1}\left(k^{\frac{1}{5}}\right)\right)-8\sin\left(2\sin^{-1}\left(k^{\frac{1}{5}}\right)\right)+12\sin^{-1}\left(k^{\frac{1}{5}}\right)$ As $k\to1$ $\sin^{-1}\left(k^{\frac{1}{5}}\right)\to\sin^{-1}(1)=\frac{\pi}{2}$ $\Rightarrow\lim_{k\to1}(f(k))=\sin(2\pi)-8\sin\pi+6\pi=6\pi$ $\Rightarrow R=\frac{5}{32}\times6\pi=\frac{15\pi}{16}$	**M1** **M1** **A1**	For use of part (b) and an upper limit of $k<1$ Integral must be found in terms of k (which could be 1) For correct use of limits
			[3]	

11.

(a)	Uses de Moivre's theorem.	1.1a	M1	By de Moivre's theorem,
	Completes a reasoned argument to obtain the required result. Must see $\cos(-n\theta)+\text{i}\sin(-n\theta)$ and $z^n - z^{-n}$	2.1	R1	$z^n = \cos n\theta + \text{i}\sin n\theta$ $z^{-n} = \cos(-n\theta)+\text{i}\sin(-n\theta)$ $= \cos(n\theta) - \text{i}\sin(n\theta)$ $z^n - z^{-n} = 2\text{i}\sin n\theta$
(b)	Uses part **(a)** to express at least three terms of S in terms of z	3.1a	M1	$2\text{i}S = 2\text{i}\sin\theta + 2\text{i}\sin 3\theta + \ldots + 2\text{i}\sin(2n-1)\theta$ $= z - z^{-1} + z^3 - z^{-3} + \ldots + z^{2n-1} - z^{-(2n-1)}$
	Expresses S or $2\text{i}S$ as the difference of two series.	1.1a	M1	$= z + z^3 + \ldots + z^{2n-1} - \left(z^{-1} + z^{-3} + \ldots + z^{-(2n-1)}\right)$
	Completes a reasoned argument to obtain the required result.	2.1	R1	$S = \frac{1}{2\text{i}}\left(z + z^3 + \ldots + z^{2n-1}\right) - \frac{1}{2\text{i}}\left(z^{-1} + z^{-3} + \ldots + z^{-(2n-1)}\right)$
(c)	Obtains expressions for the sums of their geometric series.	3.1a	M1	$2\text{i}S = \frac{z\left(1-z^{2n}\right)}{\left(1-z^2\right)} - \frac{z^{-1}\left(1-z^{-2n}\right)}{\left(1-z^{-2}\right)}$
	Obtains fully correct expressions for the sums of G_1 and G_2	1.1b	A1	$= \frac{z^{2n}-1}{z-z^{-1}} - \frac{1-z^{-2n}}{z-z^{-1}} = \frac{z^{2n}+z^{-2n}-2}{z-z^{-1}}$
	Rearranges to obtain $z - z^{-1}$ in the denominator of any fraction.	3.1a	B1	$= \frac{\left(z^n - z^{-n}\right)^2}{2\text{i}\sin\theta} = \frac{(2\text{i}\sin n\theta)^2}{2\text{i}\sin\theta} = -\frac{2\sin^2 n\theta}{\text{i}\sin\theta}$
	Obtains $z^{2n} + z^{-2n} - 2$ in the numerator of their single fraction.	3.1a	B1	$-2S = -\frac{2\sin^2 n\theta}{\sin\theta}$
	Completes a reasoned argument to obtain the required result.	2.1	R1	$S = \frac{\sin^2 n\theta}{\sin\theta}$