Topic Z2 Hyperbolics and further calculus (Post-TT A) [56] MARKSCHEME

1.

(i)		$\frac{dy}{dx}=\frac{1}{1-\left(\frac{1-x}{3+x}\right)^2}\times\frac{-(3+x)-(1-x)}{(3+x)^2}$	B1 M1 A1	Sight of standard diffn for $\tanh^{-1}x$ Fn of fn **and** diffn of quotient Soi correct quotient (i.e. correct expression for 2nd part)
		$\Rightarrow\frac{dy}{dx}=\left(\frac{-4}{(3+x)^2-(1-x)^2}\right)=\frac{k}{1+x}$	A1	
		$\Rightarrow\frac{dy}{dx}=\frac{-1}{2(1+x)}$	A1	Correct for y'
		$\Rightarrow\frac{d^2y}{dx^2}=\frac{1}{2(1+x)^2}$	A1	2nd diffn (NB AG)
			[6]	
(ii)		When $x=0, y=\tanh^{-1}\frac{1}{3}$ or $\frac{1}{2}\ln 2$ or $\ln\sqrt{2}$	B1	For 1st value (needs to be exact)
		$\frac{dy}{dx}=-\frac{1}{2}$ $\frac{d^2y}{dx^2}=\frac{1}{2}$	B1	For both
		$\Rightarrow y=\tanh^{-1}\frac{1}{3}+\left(-\frac{1}{2}\right)x+\left(\frac{1}{2}\right)\frac{x^2}{2}$	M1	Use of correct Maclaurin's series
		$=\tanh^{-1}\frac{1}{3}-\frac{1}{2}x+\frac{x^2}{4}$	A1	Accept 0.347
			[4]	

2.

5(a)	$\int\frac{1}{\sqrt{x^2+2x+10}}dx=\int\frac{1}{\sqrt{(x+1)^2+9}}dx$	M1	3.1a
	$=k\sinh^{-1}\left(\frac{x+a}{b}\right)$	M1	1.1b
	$=\sinh^{-1}\left(\frac{x+1}{3}\right)(+c)$	A1	1.1b
		(3)	
(b)	$\int_2^{20}\frac{1}{\sqrt{x^2+2x+10}}dx=\sinh^{-1}\left(\frac{20+1}{3}\right)-\sinh^{-1}\left(\frac{2+1}{3}\right)$ $=\ln\left(7+\sqrt{50}\right)-\ln\left(1+\sqrt{2}\right)=\ln\frac{7+\sqrt{50}}{1+\sqrt{2}}$	M1	1.1b
	$=\frac{1}{(20-2)}\ln\frac{7+\sqrt{50}}{1+\sqrt{2}}$	M1	2.1
	$\frac{1}{18}\ln\left(3+2\sqrt{2}\right)$ or e.g. $\frac{1}{9}\ln\left(1+\sqrt{2}\right)$	A1	2.2a
		(3)	
			(6 marks)

3.

(a)	Commences proof by considering one side of the identity only: if considering LHS combines terms as a single fraction with a common denominator. If considering RHS writes coth θ as a fraction and introduces factor of (1 + cosh θ to both numerator and denominator) Note alternative valid approaches include commencing proof by considering LHS minus RHS or LHS divided by RHS	AO2.1	R1	$\frac{\sinh\theta}{1+\cosh\theta}+\frac{1+\cosh\theta}{\sinh\theta}$ $\equiv\frac{\sinh^2\theta+1+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$ $\equiv\frac{\cosh^2\theta+\cosh^2\theta+2\cosh\theta}{(1+\cosh\theta)\sinh\theta}$, $\because 1+\sinh^2\theta\equiv\cosh^2\theta$ $\equiv\frac{2\cosh\theta(1+\cosh\theta)}{(1+\cosh\theta)\sinh\theta}$ $\equiv\frac{2\cosh\theta}{\sinh\theta}$ $\equiv 2\coth\theta$ **AG**
	Explicitly states identity $\cosh^2\theta-\sinh^2\theta\equiv 1$ and uses it to eliminate (or introduce) $\sinh^2\theta$	AO2.4	R1	
	Factorises numerator and cancels correctly for 'their' fraction (if considering RHS rearranges 'their' numerator correctly into two factorised expressions)	AO1.1b	B1F	
	Completes rigorous proof to obtain result **AG** Only award if they have a completely correct argument, which is clear and contains no slips.	AO2.1	R1	
(b)	Uses result from part **(a)** to deduce that $\tanh\theta=\frac{1}{2}$	AO2.2a	R1	$2\coth\theta=4$ $\tanh\theta=\frac{1}{2}$ $\theta=\tanh^{-1}\frac{1}{2}=\frac{1}{2}\ln 3$
	Uses natural log form and substitutes correct value to obtain correct exact form	AO1.1b	A1	
	Total		**6**	

4.

(i)	Reasonable attempt at product rule	M1	Two terms seen
	Derive or quote diff. of $\cos^{-1}x$	M1	Allow +
	Get $-x^2(1-x^2)^{-½} + (1-x^2)^{½} + (1-x^2)^{-½}$	A1	
	Tidy to $2(1-x^2)^{½}$	A1	cao
(ii)	Write down integral from (i)	B1	On any $k\sqrt{(1-x^2)}$
	Use limits correctly	M1	In any reasonable integral
	Tidy to $½\pi$	A1	
		SR	Reasonable sub. B1 Replace for new variable and attempt to integrate (ignore limits) M1 Clearly get $½\pi$ A1

5.

(i)	Express as $A/(x-1) + (Bx+C)/(x^2+9)$	M1	Allow C=0 here
	Equate (x^2+9x) to $A(x^2+9) + (Bx+C)(x-1)$	M1√	May imply above line; on their P.F.
	Sub. for x or equate coeff.	M1	Must lead to at least 3 coeff.; allow cover-up method for A
	Get A=1, B=0, C=9	A1	cao from correct method
(ii)	Get $A\ln(x-1)$	B1√	On their A
	Get $C/3 \tan^{-1}(x/3)$	B1√	On their C; condone no constant; ignore any $B \neq 0$

6.

Get $k \sinh^{-1}k_1x$	M1	For either integral; allow attempt at ln version here
Get $⅓ \sinh^{-1}¾x$	A1	Or ln version
Get $½ \sinh^{-1}⅔x$	A1	Or ln version
Use limits in their answers	M1	
Attempt to use correct ln laws to set up a solvable equation in a	M1	
Get $a = 2^{⅓}.\ 3^{½}$	A1	Or equivalent

7.

7(a)	$\int_k^8 \left((4k^2-1)y-(32k^2-k)\right)\,\mathrm{d}y=\left[(4k^2-1)\frac{y^2}{2}-(32k^2-k)y\right]_k^8$ $=(4k^2-1)\frac{8^2-k^2}{2}-(32k^2-k)(8-k)$	M1	1.1b
	$=\frac{1}{2}(4k^2-1)(8-k)(8+k)-(32k^2-k)(8-k)$ $=\frac{1}{2}(8-k)\left((4k^2-1)(8+k)-2(32k^2-k)\right)$	M1	3.1a
	$=\frac{1}{2}(8-k)\left(32k^2+4k^3-8-k-64k^2+2k\right)$ $=\frac{1}{2}(8-k)\left(4k^3-32k^2+k-8\right)$*	A1*	2.1
		(3)	
(b)	Uses $(\pi)\int x^2\,\mathrm{d}y$ with both of the curves and adds the results (a complete method to find the volume of the main body piece).	B1	3.1a
	Attempts $\int x^2\mathrm{d}y=\int\frac{y^6}{k^4}\,\mathrm{d}y=..$	M1	1.1b
	So $(\pi)\int_0^k x^2\,\mathrm{d}y=\frac{(\pi)k^3}{7}$	A1	2.2a
	Attempts second curve $\int x^2\mathrm{d}y=\int\frac{(4k^2-1)y-(32k^2-k)}{-(32-4k)}\,\mathrm{d}y=..$	M1	1.1b
	$(\pi)\int_k^8 x^2\mathrm{d}y=\frac{\frac{1}{2}(8-k)\left(4k^3-32k^2+k-8\right)}{-4(8-k)}=(\pi)\frac{1}{8}\left(8-k+32k^2-4k^3\right)$	M1	2.1
	So volume of body $=(\pi)\left(\frac{k^3}{7}+\frac{1}{8}\left(8-k+32k^2-4k^3\right)\right)$ Or total volume $=(\pi)\left(1+\frac{k^3}{7}+\frac{1}{8}\left(8-k+32k^2-4k^3\right)\right)$	A1	1.1b
	$\frac{\mathrm{d}V}{\mathrm{d}k}=0\Rightarrow\frac{3k^2}{7}+\frac{1}{8}\left(1+64k-12k^2\right)=0$	M1	3.1a
	$\Rightarrow 60k^2-448k+7=0\Rightarrow k=...$	M1	1.1b
	But $k > ½$ so must be k = awrt 7.45 cm	A1	3.2a
		(9)	