Topic Z2 Hyperbolics and further calculus (Pre-TT C) [47] MARKSCHEME

1.

$x^2+4x+8=(x+2)^2+4$	M1 A1	Complete the square in order to use standard form	
$\int_0^1 \frac{1}{\sqrt{x^2+4x+8}}\,dx=\int_0^1 \frac{1}{\sqrt{(x+2)^2+4}}\,dx$	M1	Use correct standard form in integration	
$=\left[\sinh^{-1}\frac{x+2}{2}\right]_0^1=\sinh^{-1}\left(\frac{3}{2}\right)-\sinh^{-1}1$	A1	Answer in $\sinh^{-1}$ form	
$=\ln\left(\frac{3}{2}+\sqrt{1+\frac{9}{4}}\right)-\ln\left(1+\sqrt{2}\right)=\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)-\ln\left(1+\sqrt{2}\right)$	M1	Attempt to turn into log form	
$=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	A1 [6]	www isw	
Alternative for last 4 marks $\int_0^1 \frac{1}{\sqrt{(x+2)^2+4}}\,dx=\left[\ln\left((x+2)+\sqrt{(x+2)^2+4}\right)\right]_0^1$ $=\ln\left(3+\sqrt{13}\right)-\ln\left(2+\sqrt{8}\right)=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	M1 A1 M1 A1	Attempt to use Standard form Limits www isw	
Alternative for last 4 marks $x+2=2\tan\theta \Rightarrow I=\left[\ln\left(\sec\theta+\tan\theta\right)\right]_{\pi/4}^{\tan^{-1}3/2}$ $=\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)-\ln\left(1+\sqrt{2}\right)=\ln\left(\frac{3+\sqrt{13}}{2+2\sqrt{2}}\right)$	M1 A1 M1 A1	Substitution Indefinite integral Deal with limits www isw	

2.

(i)	Attempt at correct form of P.F.	M1	Allow $Cx/(x^2+1)$ here; not $C=0$
	Rewrite as $4=$ $A(1+x)(1+x^2)+B(1-x)(1+x^2)+(Cx+D)(1-x)(1+x)$	M1 √	From their P.F.
	Use values of x/equate coefficients	M1	
	Get $A=1$, $B=1$	A1	cwo
	Get $C=0$, $D=2$	A1	
			SC Use of cover-up rule for A,B M1 If both correct A1 cwo
(ii)	Get $A\ln(1+x)-B\ln(1-x)$	M1	Or quote from List of Formulae
	Get $D\tan^{-1}x$	B1	
	Use limits in their integrated expressions	M1	
	Clearly get A.G.	A1	

3.

6	Integrate $k_1 e^{nx}$ to obtain $k_2 e^{nx}$	**M1**	any constants involving π or not; any n
	Obtain correct indefinite integral of their $k_1 e^{nx}$	**A1**	
	Substitute limits to obtain $\frac{1}{6}\pi(e^3-1)$ or $\frac{1}{6}(e^3-1)$	**A1**	or exact equiv perhaps involving e^0
	Integrate $k(2x-1)^n$ to obtain $k'(2x-1)^{n+1}$	**M1**	any constants involving π or not; any n
	Obtain correct indefinite integral of their $k(2x-1)^n$	**A1**	
	Substitute limits to obtain $\frac{1}{18}\pi$ or $\frac{1}{18}$	**A1**	or exact equiv
	Apply formula $\int \pi y^2\,dx$ at least once	**B1**	for $y=e^{3x}$ and/or $y=(2x-1)^4$
	Subtract, correct way round, attempts at volumes	**M1**	allow with π missing but must involve y^2
	Obtain $\frac{1}{6}\pi e^3-\frac{2}{9}\pi$	**A1** **9**	or similarly simplified exact equiv

4.

1	**(i)**		$\left(e^x+e^{-x}\right)^3=e^{3x}+3e^x+3e^{-x}+e^{-3x}$	**M1**	Doing the expansion	
			$=\left(e^{3x}+e^{-3x}\right)+3\left(e^x+e^{-x}\right)$	**A1**		
			$\Rightarrow\left(2\cosh x\right)^3=2\cosh 3x+6\cosh x$	**M1**	Relating cosh3*x* to exponentials correctly	
			$\Rightarrow 8\cosh^3 x=2\cosh 3x+6\cosh x$	**A1**		
			$\Rightarrow \cosh 3x=4\cosh^3 x-3\cosh x$	**[4]**		
	(ii)		$\Rightarrow \cosh 3x=4\cosh^3 x-3\cosh x=6\cosh x$	**M1**	Using result of (i)	
			$\Rightarrow 4\cosh^3 x=9\cosh x$			
			$\Rightarrow \cosh^2 x=\frac{9}{4}$ since $\cosh x\neq 0$	**A1**	At least one rejection needs to be stated. Or cosh*x* ≥ 1	
			$\Rightarrow \cosh x=(\pm)\frac{3}{2}$ $\cosh x\neq -\frac{3}{2}$	**A1**		
			$\Rightarrow x=\pm\ln\left(\frac{3}{2}+\sqrt{\left(\frac{3}{2}\right)^2-1}\right)$	**A1** **A1**	A1 for each in exact form Deduct from 5 marks 1 mark for additional incorrect answers	
			$=\pm\ln\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)$ or $\ln\left(\frac{3}{2}\pm\frac{1}{2}\sqrt{5}\right)$	**[5]**		
			Alternative:	**M1**	Using exponentials	
			$\cosh 3x=6\cosh x\Rightarrow\frac{1}{2}\left(e^{3x}+e^{-3x}\right)=3\left(e^x+e^{-x}\right)$			
			$\Rightarrow e^{3x}-6e^x-6e^{-x}+e^{-3x}=0\Rightarrow e^{6x}-6e^{4x}-6e^{2x}+1=0$			
			let $y=e^{2x}$			
			$\Rightarrow y^3-6y^2-6y+1=0\Rightarrow\left(y+1\right)\left(y^2-7y+1\right)=0$	**A1**	Cubic in factorised form	
			$\Rightarrow y=-1,\frac{7\pm\sqrt{45}}{2}$	**A1**	Rejection of $y=-1$ must be stated.	
			$e^{2x}\neq-1\Rightarrow e^{2x}=\frac{7\pm\sqrt{45}}{2}\Rightarrow x=\frac{1}{2}\ln\left(\frac{7\pm\sqrt{45}}{2}\right)=\frac{1}{2}\ln\left(\frac{7\pm3\sqrt{5}}{2}\right)$	**A1** **A1**	**oe** in exact form Deduct from 5 marks 1 mark for additional incorrect answers	

5.

5(a)	$\frac{dy}{dx}=\frac{1}{\sinh^2 x+1}\times ..$	M1	1.2
	$\frac{dy}{dx}=\frac{1}{\sinh^2 x+1}\times\cosh x$	A1	1.1b
	$=\frac{\cosh x}{\cosh^2 x}=\operatorname{sech} x$ or use of correct identity $\sinh^2 x + 1 = \cosh^2 x$ later in the proof.	B1	2.1
	E.g. $\frac{d^2y}{dx^2}=-\operatorname{sech} x\tanh x$ or $\frac{d^2y}{dx^2}=-(\cosh x)^{-2}\times\sinh x$ or even $\frac{d^2y}{dx^2}=\frac{(\sinh x)(\sinh^2 x+1)-(\cosh x)(2\sinh x\cosh x)}{\left(\sinh^2 x+1\right)^2}$	M1	1.1b
	$\frac{d^3y}{dx^3}=-(-\operatorname{sech} x\tanh x)(\tanh x)+(-\operatorname{sech} x)(\operatorname{sech}^2 x)$ (oe) or any valid attempt at the third derivative from their second derivative. E.g. $\frac{d^2y}{dx^2}=-\tanh x\frac{dy}{dx}$ then $\frac{d^3y}{dx^3}=-\operatorname{sech}^2 x\frac{dy}{dx}-\tanh x\frac{d^2y}{dx^2}$	M1 A1	3.1a 1.1b
	E.g. $\frac{d^3y}{dx^3}=\operatorname{sech} x\tanh^2 x-\operatorname{sech}^3 x=\operatorname{sech} x(1-\operatorname{sech}^2 x)-\operatorname{sech}^3 x$ $=\operatorname{sech} x-2\operatorname{sech}^3 x=\frac{dy}{dx}-2\left(\frac{dy}{dx}\right)^3$ * **or** $\frac{d^3y}{dx^3}=-\operatorname{sech}^2 x\frac{dy}{dx}-\tanh x\frac{d^2y}{dx^2}=-\left(\frac{dy}{dx}\right)^3+\tanh^2 x\frac{dy}{dx}$ $=(1-\operatorname{sech}^2 x)\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^3=\frac{dy}{dx}-2\left(\frac{dy}{dx}\right)^3$ *	A1*	2.1
(b)	$\frac{d^4y}{dx^4}=\frac{d^2y}{dx^2}-6\left(\frac{dy}{dx}\right)^2\times\frac{d^2y}{dx^2}$	M1 A1	1.1b 1.1b
	$\frac{d^5y}{dx^5}=\frac{d^3y}{dx^3}-12\left(\frac{dy}{dx}\right)\times\left(\frac{d^2y}{dx^2}\right)^2-6\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}$	M1 A1	2.1 1.1b
(c)	At $x=0, y=0, y'=1, y''=0, y^{(3)}=-1, y^{(4)}=0$ and $y^{(5)}=-1-1\times0^2-6\times1^2\times(-1)=5$	M1	1.1b
	So $y=y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y^{(3)}(0)+\frac{x^4}{4!}y^{(4)}(0)+\frac{x^5}{5!}y^{(5)}(0)+\ldots$ with their evaluated values.	M1	1.1b
	$y=x-\frac{x^3}{6}+\frac{x^5}{24}+\ldots$	A1	2.5