Topic Z3 Differential equations (Post-TT A) [53]

1.

(i) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4x = 0,$$

where k is a real constant, in each of the following cases.

(a) $|k| > 2$

(b) $|k| < 2$

(c) $k = 2$

[8]

(ii) (a) In the case when $k = 1$, find the solution for which $x = 0$ and $\frac{dx}{dt} = 6$ when $t = 0$. [4]

(b) Describe what happens to x as $t \to \infty$ in this case, justifying your answer. [2]

(Total 14 marks)

2.

Find the solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x$$

for which $y = 2$ when $x = \frac{1}{6}\pi$. Give your answer in the form $y = f(x)$. [9]

(Total 9 marks)

3.

A damped spring is part of a car suspension system. In tests for the system, a mass is attached to the damped spring and is made to move upwards in a vertical line.

The motion of the system is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 2e^{-3t}$$

where x cm is the vertical displacement of the mass above its equilibrium position and t is the time, in seconds, after motion begins.

In one particular test, the mass is moved to a position 20 cm above its equilibrium position and given an initial velocity of 1 ms^{-1} upwards. For this test, use the model to

(a) find an equation for x in terms of t,

(9)

(b) find, to the nearest mm, the maximum displacement of the mass from its equilibrium position.

(3)

In this test, the time taken for the mass to return to its equilibrium position was measured as 2.86 seconds.

(c) State, with justification, whether or not this supports the model.

(1)

(Total 13 marks)

4.

A doctor is studying the concentration of an antibiotic in the blood and the body tissue of a patient.
Let x be the number of micrograms of the antibiotic in the blood.
Let y be the number of micrograms of the antibiotic in the body tissue.
The doctor models her results by the differential equations

$$\frac{dx}{dt} = -5x + y + 51$$

$$\frac{dy}{dt} = 12x - 6y$$

where t is the time in hours after a dose of the antibiotic has been administered to the patient.

(a) Show that

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 18x = 306$$

(3)

(b) Find a general solution for the number of micrograms of the antibiotic in the blood at time t hours.

(6)

(c) Hence find a general solution for the number of micrograms of the antibiotic in the body tissue at time t hours.

(2)

Initially there is none of this antibiotic in the blood and none of this antibiotic in the body tissue.

(d) Find, in minutes, to 2 decimal places, the time when the rate of increase of the antibiotic in the blood is equal to the rate of increase of the antibiotic in the body tissue.

(5)

(e) Evaluate the model.

(1)

(Total 17 marks)