Topic Z3 Differential equations (Post-TT A) [53] MARKSCHEME

1.

(i) $m^2+2km+4=0$	M1	For stating and attempting to solve auxiliary eqn
$\Rightarrow m=-k\pm\sqrt{k^2-4}$	A1 **2**	For correct solutions, at any stage **AEF**
(a) $x=\mathrm{e}^{-kt}\left(A\mathrm{e}^{\sqrt{k^2-4}\,t}+B\mathrm{e}^{-\sqrt{k^2-4}\,t}\right)$	M1 A1 **2**	For using $\mathrm{e}^{\mathrm{f}(t)}$ with distinct real roots of aux eqn For correct answer **AEF**
(b) $x=\mathrm{e}^{-kt}\left(A\mathrm{e}^{\mathrm{i}\sqrt{4-k^2}\,t}+B\mathrm{e}^{-\mathrm{i}\sqrt{4-k^2}\,t}\right)$	M1	For using $\mathrm{e}^{\mathrm{f}(t)}$ with complex roots of aux eqn This form may not be seen explicitly but if stated as final answer earns M1 A0
$x=\mathrm{e}^{-kt}\left(A'\cos\sqrt{4-k^2}\,t+B'\sin\sqrt{4-k^2}\,t\right)$ *OR* $x=\mathrm{e}^{-kt}\left(C'\begin{matrix}\cos\\\sin\end{matrix}\left(\sqrt{4-k^2}\,t+\alpha\right)\right)$	A1 **2**	For correct answer
(c) $x=\mathrm{e}^{-2t}\left(A''+B''t\right)$	M1 A1 **2**	For using $\mathrm{e}^{\mathrm{f}(t)}$ with equal roots of aux eqn For correct answer. Allow k for 2
(ii)(a) $x=B'\mathrm{e}^{-t}\sin\sqrt{3}\,t$	B1 √	For using $t=0, x=0$ correctly. f.t. from **(b)**
$\dot{x}=B'\mathrm{e}^{-t}\left(\sqrt{3}\cos\sqrt{3}\,t-\sin\sqrt{3}\,t\right)$	M1 A1 √	For differentiating x For correct expression. f.t. from their x
$t=0, \dot{x}=6\Rightarrow B'=2\sqrt{3}$, $x=2\sqrt{3}\mathrm{e}^{-t}\sin\sqrt{3}\,t$	A1 **4**	For correct solution **AEF** **SR** √ and **AEF** OK for $x=C'\mathrm{e}^{-t}\cos\left(\sqrt{3}\,t+\frac{1}{2}\pi\right)$
(b) $x\to 0$	B1	For correct statement
$\mathrm{e}^{-t}\to 0$ and sin( ) is bounded	B1 **2** **14**	For both statements

2.

Integrating factor $=\mathrm{e}^{\int\cot x\,\mathrm{d}x}=\mathrm{e}^{\ln\sin x}=\sin x$	M1 A1	For IF $=\mathrm{e}^{\pm\ln\sin x}$ *OR* $\mathrm{e}^{\pm\ln\cos x}$ For simplified IF	
$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}(y\sin x)=2x\sin x$	M1	For $\frac{\mathrm{d}}{\mathrm{d}x}(y.\text{their IF})=2x.\text{their IF}$	
$\Rightarrow y\sin x=-2x\cos x+\int 2\cos x\,\mathrm{d}x$	M1*	For attempt to integrate RHS using parts for $\int x\begin{cases}\sin x\\\cos x\end{cases}\mathrm{d}x$	(Must use u = (2)x)
$\Rightarrow y\sin x=-2x\cos x+2\sin x\ (+c)$	A1 A1	For correct RHS 1st stage oe	
$\left(\frac{1}{6}\pi, 2\right)\Rightarrow c=\frac{1}{6}\pi\sqrt{3}$	M1dep *	For substituting $\left(\frac{1}{6}\pi, 2\right)$ into their GS (with c)	c = 0.907
$\Rightarrow y=-2x\cot x+2+\frac{1}{6}\pi\sqrt{3}\operatorname{cosec}x$	A1 FT A1 [9]	For correctly finding c (FT from GS) For correct solution **AEF** of standard notation $y=\mathrm{f}(x)$	

3.

6(a)	$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=2e^{-3t}$		
	AE: $m^2+6m+9=0 \Rightarrow (m+3)^2=0 \Rightarrow m=\ldots\ (=-3)$	M1	1.1b
	So C.F. is $x_{CF}=(A+Bt)e^{-3t}$	A1	2.2a
	For P.I. try $x_{PI}=kt^2e^{-3t}$	B1	2.2a
	$\dot{x}_{PI}=2kte^{-3t}-3kt^2e^{-3t}\quad (=k(2t-3t^2)e^{-3t})$ $\ddot{x}_{PI}=2ke^{-3t}-6kte^{-3t}-6kte^{-3t}+9kt^2e^{-3t}\quad (=k(2-12t+9t^2)e^{-3t})$ $\Rightarrow k(2-12t+9t^2)e^{-3t}+6k(2t-3t^2)e^{-3t}+9kt^2e^{-3t}=2e^{-3t} \Rightarrow k=\ldots$	M1	1.1b
	So $k=1$ ie $x_{PI}=t^2e^{-3t}$	A1	1.1b
	General solution is $x=(A+Bt)e^{-3t}+t^2e^{-3t}$ (their C.F. + their P.I.)	M1	1.1a
	$x(0)=20 \Rightarrow A=20$	M1	3.4
	$\dot{x}=Be^{-3t}-3(A+Bt)e^{-3t}+2te^{-3t}-3t^2e^{-3t}=(B-3A+(2-3B)t-3t^2)e^{-3t}$ $x(0)=100 \Rightarrow B=100+3A=\ldots(=160)$	M1	3.4
	So $x=(20+160t+t^2)e^{-3t}$	A1	1.1b
		(9)	
(b)	From above $x=(B-3A+(2-3B)t-3t^2)e^{-3t}=(100-478t-3t^2)e^{-3t}$		
	$x=0 \Rightarrow 100-478t-3t^2=0 \Rightarrow t=..\quad (=-159.5\ldots \text{ or } 0.2089\ldots)$	M1	3.1a
	$t>0$, so $t_{max}=0.2089\ldots \Rightarrow$ $x_{max}=\left(20+160\times0.2089\ldots+(0.2089\ldots)^2\right)e^{-3\times0.2089\ldots}=\ldots$	M1	3.4
	$x_{max}=\text{awrt } 28.6\text{ cm }(3\text{ s.f.})$ (28.57055381741878)	A1	1.1b
		(3)	
(c)	$x(2.86)=0.0912\ldots$ which is close to zero (less than 1mm), which can be accounted for by inaccuracies in measurements. So the model is supported by this measurement.	B1ft	2.2b
		(1)	
			(13 marks)

4.

8(a)	$y=\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5\frac{\mathrm{d}x}{\mathrm{d}t}$	B1	2.1
	$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5\frac{\mathrm{d}x}{\mathrm{d}t}=12x-6\left(\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51\right)$	M1	2.1
	$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+11\frac{\mathrm{d}x}{\mathrm{d}t}+18x=306*$	A1*	1.1b
		(3)	
(b)	$m^2+11m+18=0 \Rightarrow m=...$	M1	3.4
	$m=-2,-9$	A1	1.1b
	$x=A\mathrm{e}^{\alpha t}+B\mathrm{e}^{\beta t}$	M1	3.4
	$x=A\mathrm{e}^{-9t}+B\mathrm{e}^{-2t}$	A1	1.1b
	PI: Try $x=k \Rightarrow 18k=306$ $\Rightarrow k=17$	M1	3.4
	$GS: x=A\mathrm{e}^{-9t}+B\mathrm{e}^{-2t}+17$	A1ft	1.1b
		(6)	
(c)	$y=\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51 \Rightarrow y=-9A\mathrm{e}^{-9t}-2B\mathrm{e}^{-2t}+5A\mathrm{e}^{-9t}+5B\mathrm{e}^{-2t}+85-51$	M1	3.4
	$y=3B\mathrm{e}^{-2t}-4A\mathrm{e}^{-9t}+34$	A1	1.1b
		(2)	
(d)	$0=A+B+17,\ 0=3B-4A+34 \Rightarrow A=...,B=...$ (NB $A=-\frac{17}{7}, B=-\frac{102}{7}$)	M1	3.3
	$x=17-\frac{17}{7}\mathrm{e}^{-9t}-\frac{102}{7}\mathrm{e}^{-2t}, y=34+\frac{68}{7}\mathrm{e}^{-9t}-\frac{306}{7}\mathrm{e}^{-2t}$	A1	1.1b
	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{153}{7}\mathrm{e}^{-9t}+\frac{204}{7}\mathrm{e}^{-2t}=-\frac{612}{7}\mathrm{e}^{-9t}+\frac{612}{7}\mathrm{e}^{-2t} \Rightarrow \mathrm{e}^{k}=\alpha$	M1	3.1b
	$\mathrm{e}^{7t}=\frac{15}{8} \Rightarrow 7t=\ln\left(\frac{15}{8}\right) \Rightarrow t=\frac{1}{7}\ln\left(\frac{15}{8}\right)$	M1	1.1b
	= 5.39 minutes	A1	3.2a
		(5)	
(e)	E.g. • The model suggests that, in the long term, the amount of antibiotic in the blood (and/or the body tissue) will remain constant and this is unlikely	B1	3.5a
		(1)	
			(17 marks)