Topic Z3 Differential equations (Pre-TT A) [51] MARKSCHEME

1.

Question	Scheme	Marks	AOs
5(a)	$(t+4)\frac{\mathrm{d}v}{\mathrm{d}t}+5v=10(t+4)\Rightarrow\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{5v}{(t+4)}=10$	M1	1.1b
	$\text{IF}=\mathrm{e}^{\int\frac{5}{t+4}\mathrm{d}t}=(t+4)^5\Rightarrow v(t+4)^5=\int 10(t+4)^5\mathrm{d}t$	M1	3.1b
	$v(t+4)^5=\frac{5}{3}(t+4)^6+c$	A1	1.1b
	$t=0,\ v=0\Rightarrow c=-\frac{20480}{3}$	M1	3.4
	$t=3\Rightarrow v=\frac{5}{3}\times 7-\frac{20480}{3\times 7^5}$	M1	3.4
	$v=11.3\left(\text{ms}^{-1}\right)$	A1	1.1b
		(6)	
(b)	For large values of t, the velocity increases	B1	1.1b
		(1)	
(c)	E.g. • The raindrop may hit an obstacle as it falls • The raindrop is unlikely to be at rest initially • The raindrop may be affected by the wind as it falls • The raindrop will eventually hit the ground	B1	3.5b
		(1)	
			(8 marks)

Notes

(a)
M1: Divides through by $(t + 4)$
M1: Uses the model to find the integrating factor and attempts the solution of the differential equation
A1: Correct solution
M1: Interprets the initial conditions to find the constant of integration
M1: Uses their solution to the problem to find the velocity after 3 seconds
A1: Correct value
(b)
B1: Makes a sensible comment regarding the motion of the raindrop e.g. as t increases so does v
(c)
B1: States a limitation of the model – see scheme for examples

2.

(a)	$r = 10\frac{df}{dt} - 2f \Rightarrow \frac{dr}{dt} = 10\frac{d^2 f}{dt^2} - 2\frac{df}{dt}$	M1	2.1
	$10\frac{d^2 f}{dt^2} - 2\frac{df}{dt} = -0.2f + 0.4\left(10\frac{df}{dt} - 2f\right)$	M1	2.1
	$\frac{d^2 f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$*	A1*	1.1b
		(3)	
b)	$m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4 \times 0.1}}{2}$	M1	3.4
	$m = 0.3 \pm 0.1\text{i}$	A1	1.1b
	$f = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$	M1	3.4
	$f = e^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	A1	1.1b
		(4)	
c)	$\frac{df}{dt} = 0.3e^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right) + 0.1e^{0.3t}\left(B\cos 0.1t - A\sin 0.1t\right)$	M1	3.4
	$r = 10\frac{df}{dt} - 2f$ $= e^{0.3t}\left((3A+B)\cos 0.1t + (3B-A)\sin 0.1t\right) - 2e^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	M1	3.4
	$r = e^{0.3t}\left((A+B)\cos 0.1t + (B-A)\sin 0.1t\right)$	A1	1.1b
		(3)	
d)(i)	$t = 0, f = 6 \Rightarrow A = 6$	M1	3.1b
	$t = 0, r = 20 \Rightarrow B = 14$	M1	3.3
	$r = e^{0.3t}\left(20\cos 0.1t + 8\sin 0.1t\right) = 0$	M1	3.1b
	$\tan 0.1t = -2.5$	A1	1.1b
	2019	A1	3.2a
d)(ii)	3750 foxes	B1	3.4
d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible	B1	3.5a
		(7)	
			(17 marks)

3.

Question	Scheme	Marks	AOs
8(a)(i)	Container contains 3+0.25t −0.125t = 3 + 0.125t litres after t minutes	B1	3.3
(ii)	Rate of contaminant out = $0.125\times\dfrac{m}{3+0.125t}$ mg per minute	M1	3.3
	Rate of contaminant in = 0.25 × (5−$e^{-0.1t}$) mg per minute	B1	2.2a
	$\dfrac{\mathrm{d}m}{\mathrm{d}t}=\dfrac{5-\mathrm{e}^{-0.1t}}{4}-\dfrac{m}{24+t}$ *	A1*	1.1b
		(4)	
(b)	Rearranges to form $\dfrac{\mathrm{d}m}{\mathrm{d}t}+\dfrac{m}{24+t}=\dfrac{5-\mathrm{e}^{-0.1t}}{4}$ and attempts integrating factor (may be by recognition).	M1	3.1a
	I.F. = $\left(\mathrm{e}^{\int\frac{1}{24+t}\,\mathrm{d}t}=\mathrm{e}^{\ln(24+t)}\right)=24+t$	A1	1.1b
	$(24+t)m=\dfrac{1}{4}\int(24+t)(5-\mathrm{e}^{-0.1t})\,\mathrm{d}t=\dfrac{1}{4}\int 120+5t-24\mathrm{e}^{-0.1t}-t\mathrm{e}^{-0.1t}\,\mathrm{d}t=..$	M1	3.1a
	$=\dfrac{1}{4}\left(120t+\dfrac{5t^2}{2}-\dfrac{24\mathrm{e}^{-0.1t}}{-0.1}+\ldots\right)$	A1	1.1b
	$\int t\mathrm{e}^{-0.1t}\,\mathrm{d}t=t\dfrac{\mathrm{e}^{-0.1t}}{-0.1}-\int 1\times\dfrac{\mathrm{e}^{-0.1t}}{-0.1}\,\mathrm{d}t=t\dfrac{\mathrm{e}^{-0.1t}}{-0.1}-\dfrac{\mathrm{e}^{-0.1t}}{(-0.1)^2}$	M1 A1	1.1b 1.1b
	So $(24+t)m=\dfrac{5}{8}t^2+30t+85\mathrm{e}^{-0.1t}+\dfrac{5}{2}t\mathrm{e}^{-0.1t}+c$		
	When t = 0, m = 0 as initially no contaminant in the container, so 0 = 0 + 0 + 85 + 0 + c ⇒ c = −85	M1	3.4
	$m=\dfrac{1}{24+t}\left(\dfrac{5}{8}t^2+30t+85\mathrm{e}^{-0.1t}+\dfrac{5}{2}t\mathrm{e}^{-0.1t}-85\right)$	A1	2.2b
		(8)	
(c)	When t = 30 m = 25.65677... and V = 6.75, hence the concentration is 3.80 mg per litre.	M1	3.4
	This resembles the measured value very closely and could easily be explained by minor inaccuracies in measurements, so the model seems to be suitable over this timeframe.	A1	3.5a
		(2)	
			(14 marks)

4.

(a)(i)	Weight = mass × g $\Rightarrow m = \dfrac{30000}{g} = 3000$ But mass is in thousands of kg, so $m = 3$	M1	3.3
(ii)	$\dfrac{dx}{dt} = 40\cos t + 20\sin t,\ \dfrac{d^2x}{dt^2} = -40\sin t + 20\cos t$	M1	1.1b
	$3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t)$ $+ 40\sin t - 20\cos t = \ldots$	M1	1.1b
	$= 200\cos t$ so PI is $x = 40\sin t - 20\cos$	A1*	2.1
	or		
	Let $x = a\cos t + b\sin t$ $\dfrac{dx}{dt} = -a\sin t + b\cos t,\ \dfrac{d^2x}{dt^2} = -a\cos t - b\sin t$	M1	1.1b
	$4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$	M1	2.1
	$x = 40\sin t - 20\cos t$	A1*	1.1b
(iii)	$3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\dfrac{1}{3}$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t}$	A1	1.1b
	$x = PI + CF$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
		(8)	
(b)	$t = 0, x = 0 \Rightarrow A + B = 20$	M1	3.4
	$x = 0, \dfrac{dx}{dt} = -Ae^{-t} - \dfrac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ $\Rightarrow A + \dfrac{1}{3}B = 40$	M1	3.4
	$x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
	$t = 9 \Rightarrow x = 33\text{m}$	A1	3.4
		(4)	
			(12 marks)