Topic Z3 Differential equations (Pre-TT B) [51]

1. (a) Obtain the general solution of the differential equation

$$\tan x \frac{dy}{dx} + y = \sin x \tan x$$

where $0 < x < \frac{\pi}{2}$.

[5]

(b) Hence find the particular solution of this differential equation, given that $y = \frac{1}{2\sqrt{2}}$ when $x = \frac{\pi}{4}$.

[2]

(Total 10 marks)

2.

The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}.$$

(i) Find the complementary function. **[3]**

(ii) Explain briefly why there is no particular integral of either of the forms $y = ke^{3x}$ or $y = kxe^{3x}$. **[1]**

(iii) Given that there is a particular integral of the form $y = kx^2e^{3x}$, find the value of k. **[5]**

(Total marks)

3.

A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{dx}{dt} = 50 - \frac{4x}{200 + t}$$

(4)

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(5)

(c) Explain how the model could be refined.

(1)

(Total 10 marks)

4. A small, hollow, plastic ball, of mass m kg is at rest on a point O on a polished horizontal surface. The ball is attached to two identical springs. The other ends of the springs are attached to the points P and Q which are 1.8 metres apart on a straight line through O.

The ball is struck so that it moves away from O, towards P with speed of 0.75 ms^{-1}.
As the ball moves, its displacement from O is x metres at time t seconds after the motion starts.
The force that each of the springs applies to the ball is $12.5\,mx$ newtons towards O.

The ball is to be modelled as a particle. The surface is assumed to be smooth and it is assumed that the forces applied to the ball by the springs are the only horizontal forces acting on the ball.

(a) Find the minimum distance of the ball from P, in the subsequent motion.

[5]

(b) In practice the minimum distance predicted by the model is incorrect. Is the minimum distance predicted by the model likely to be too big or too small? Explain with reference to the model. [2]

(Total 7 marks)

5. The watering of crops on a farm is thought to affect the concentration of nitrates in a nearby river. In a study, the concentration of nitrates in the river is measured at a point downstream from the farm.
The concentration of nitrates is modelled by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 1$$

where y is the concentration in milligrams per litre, t hours after the crops were watered.

(a) Find a general solution for the concentration of nitrates after time t hours.

(6)

Initially

- the concentration of nitrates was measured as 1 milligram per litre,
- according to the model, the concentration was increasing at a rate of 9 milligram per litre every hour.

(b) Find the particular solution for the concentration of nitrates after t hours.

(3)

(c) Hence determine the maximum concentration of nitrates after the crops are watered.

(3)

The concentration of nitrates is believed to return to its initial concentration 8 hours after the crops are watered.

(d) State, with justification, whether this is supported by the model.

(2)

(Total 15 marks)