Topic Z3 Differential equations (Pre-TT B) [51] MARKSCHEME

1.

6(a)	Selects appropriate method for example by changing to reduced equation by dividing by tan x	AO3.1a	M1	$\frac{dy}{dx}+(\cot x)y=\sin x$
	Finds correct integrating factor	AO1.1b	B1	Integrating factor $=e^{\int(\cot x)dx}$ $=e^{\ln(\sin x)}=\sin x$
ALT	Alt. finds an integrating factor by inspection, using original equation. (PI)	AO3.1a	M1	$\cos x\tan x\frac{dy}{dx}+y\cos x=\sin x\tan x\cos x$ $\sin x\frac{dy}{dx}+(\cos x)y=\sin^2 x$
	finds integrating factor = cos x	AO1.1b	B1	
	Multiples reduced or original equation by 'their' integrating factor and identifies LHS as differential of $y\times\sin x$ PI	AO1.1a	M1	$\sin x\frac{dy}{dx}+(\cos x)y=\sin^2 x$ $\frac{d}{dx}[y\sin x]=\sin^2 x$
	Uses appropriate integration method for RHS of 'their' equation	AO1.1a	M1	$y\sin x=\int\left\{\frac{1}{2}(1-\cos 2x)\right\}dx$
	Integrates correctly to obtain correct solution	AO1.1b	A1	$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$
(b)	Uses boundary condition after integration completed in either $y\sin x=\ldots$ or $y=\ldots$ form OE	AO1.1a	M1	$\sin\frac{\pi}{4}.\frac{1}{2\sqrt{2}}=\frac{1}{2}.\frac{\pi}{4}-\frac{1}{4}\sin\frac{\pi}{2}+C$ $C=\frac{1}{2}-\frac{\pi}{8}$
	States fully correct particular solution	AO1.1b	A1	$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{1}{2}-\frac{\pi}{8}$
	Total		**7**	

2.

(i)	$m^2-6m+9\ (=0) \Rightarrow m=3$	M1 A1		For attempting to solve correct auxiliary equation For correct m
	$\text{CF} = (A+Bx)e^{3x}$	A1	3	For correct CF
(ii)	ke^{3x} and kxe^{3x} both appear in CF	B1	1	For correct statement
(iii)	$y=kx^2e^{3x} \Rightarrow y'=2kxe^{3x}+3kx^2e^{3x}$	M1 A1		For differentiating kx^2e^{3x} twice For correct y' aef
	$\Rightarrow y''=2ke^{3x}+12kxe^{3x}+9kx^2e^{3x}$	A1		For correct y'' aef
	$\Rightarrow ke^{3x}\left(2+12x+9x^2-12x-18x^2+9x^2\right)=e^{3x}$	M1		For substituting y'', y', y into DE
	$\Rightarrow k=\frac{1}{2}$	A1	5	For correct k
		9		

3.

(a)	Pond contains 1000 + 5t litres after t days	M1	3.3
	If x is the amount of pollutant in the pond after t days Rate of pollutant out = $20\times\dfrac{x}{1000+5t}$ g per day	M1	3.3
	Rate of pollutant in = 25 × 2 g = 50g per day	B1	2.2a
	$\dfrac{dx}{dt}=50-\dfrac{4x}{200+t}$ *	A1*	1.1b
		(4)	
b)	$I=e^{\int\frac{4}{200+t}dt}=(200+t)^4 \Rightarrow x(200+t)^4=\int 50(200+t)^4\,dt$	M1	3.1b
	$x(200+t)^4=10(200+t)^5+c$	A1	1.1b
	$x=0,\ t=0 \Rightarrow c=-3.2\times10^{12}$	M1	3.4
	$t=8 \Rightarrow x=10(200+8)-\dfrac{3.2\times10^{12}}{(200+8)^4}$	M1	1.1b
	$=370$g	A1	2.2b
		(5)	
c)	e.g. • The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry • The rate of leaking could be made to vary with the volume of water in the pond	B1	3.5c
		(1)	
			(10 marks)

4.

(a)	Models the motion of the ball by forming an equation of motion	AO3.1b	M1	$m\frac{d^2x}{dt^2}=(-12.5mx)\times 2$ $\Rightarrow \frac{d^2x}{dt^2}=-25x$
	Uses SHM equations to form model for displacement	AO3.1a	M1	$\therefore x = A\sin(5t)$
	Uses initial condition to find the constant	AO1.1a	M1	$\Rightarrow \dot{x}=5A\cos(5t)$
	Obtains correct value for constant	AO1.1b	A1	when $t=0, \dot{x}=0.75$ so $0.75=5A$ $A=0.15$
	Interprets 'their' value to find minimum distance from P	AO3.2a	A1F	Hence $x=0.15\sin(5t)$ Max displacement = 0.15 metres from O, when $\sin(5t)=\pm 1$, so minimum distance from P is 0.75 metres
(b)	Identifies a correct limitation of the model for example friction between ball and the surface or damping effect due to air	AO3.5b	B1	It is unlikely that the surface is perfectly smooth so friction will be acting. The ball will be likely to travel a smaller distance before coming to rest and the minimum distance of the ball from P may actually be greater than that calculated in part **(a)**.
	Correctly infers whether the distance is too big or too small based on the limitation they have identified. Accept any well-reasoned inference.	AO2.2b	R1	
	Total		**7**	

5.

Question	Scheme	Marks	AOs
9(a)	$m^2+2m+1=0\Rightarrow m=...(-1)$	M1	1.1b
	$CF: y=(At+B)\text{e}^{-t}$	M1	2.2a
	PI : Try $y=kt^2\text{e}^{-t}+c$	B1	2.2a
	$\dfrac{\text{d}y}{\text{d}t}=2kt\text{e}^{-t}-kt^2\text{e}^{-t},\ \dfrac{\text{d}^2y}{\text{d}t^2}=2k\text{e}^{-t}-2kt\text{e}^{-t}+kt^2\text{e}^{-t}-2kt\text{e}^{-t}$ $2k\text{e}^{-t}-4kt\text{e}^{-t}+kt^2\text{e}^{-t}+2\left(2kt\text{e}^{-t}-kt^2\text{e}^{-t}\right)+kt^2\text{e}^{-t}+1=\text{e}^{-t}+1\Rightarrow k=...$	M1	1.1b
	$k=\dfrac{1}{2}\Rightarrow PI: y=\dfrac{1}{2}t^2\text{e}^{-t}+1$	A1	1.1b
	$y=CF+PI=(At+B)\text{e}^{-t}+\dfrac{1}{2}t^2\text{e}^{-t}+1$	A1	1.1a
		(6)	
(b)	$t=0,\ y=1\Rightarrow B=0$	M1	3.4
	$y=\left(At+0.5t^2\right)\text{e}^{-t}+1\Rightarrow\dfrac{\text{d}y}{\text{d}t}=\left(A+t-At-0.5t^2\right)\text{e}^{-t}$ $t=0,\ \dfrac{\text{d}y}{\text{d}t}=9\Rightarrow A=9$	M1	3.4
	$y=\left(0.5t^2+9t\right)\text{e}^{-t}+1$	A1	1.1b
		(3)	
(c)	$\dfrac{\text{d}y}{\text{d}t}=0\Rightarrow 0.5t^2+8t-9=0\Rightarrow t=...$	M1	3.1a
	$t>0\Rightarrow t=1.0553...\Rightarrow y=...$	M1	3.4
	$y=4.50\text{mg}/\text{l}$	A1	1.1b
		(3)	
(d)	$t=8\Rightarrow y=\left(0.5\times8^2+9\times8\right)\text{e}^{-8}+1=1.03488...$ • This is close to 1 so the model supports the suggestion that the concentration returns to its initial value after around 8 hours	M1 A1ft	3.4 3.2b
		(2)	
			(14 marks)