Topic Z4 Complex numbers (Post-TT A) [57]

1.

(a) Given that $z_1 = 2e^{\frac{1}{6}\pi i}$ and $z_2 = 3e^{\frac{1}{4}\pi i}$, express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $re^{i\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [4]

(b) Given that $w = 2(\cos\frac{1}{8}\pi + i\sin\frac{1}{8}\pi)$, express w^{-5} in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [3]

(Total 7 marks)

2.

In this question, w denotes the complex number $\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$.

(i) Express w^2, w^3 and w^* in polar form, with arguments in the interval $0 \leqslant \theta < 2\pi$. **[4]**

(ii) The points in an Argand diagram which represent the numbers

$$1, \quad 1+w, \quad 1+w+w^2, \quad 1+w+w^2+w^3, \quad 1+w+w^2+w^3+w^4$$

are denoted by A, B, C, D, E respectively. Sketch the Argand diagram to show these points and join them in the order stated. (Your diagram need not be exactly to scale, but it should show the important features.) **[4]**

(iii) Write down a polynomial equation of degree 5 which is satisfied by w. **[1]**

(Total 9 marks)

3.

(i) By expressing $\cos\theta$ in terms of $e^{i\theta}$ show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$ **[4]**

(ii) Hence solve, for $0 \leqslant \theta \leqslant \pi$,

$$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3.$$ **[5]**

(Total 9 marks)

4.

(i) Use de Moivre's theorem to find an expression for $\tan 4\theta$ in terms of $\tan\theta$. [4]

(ii) Deduce that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. [1]

(iii) Hence show that one of the roots of the equation $x^2 - 6x + 1 = 0$ is $\cot^2(\frac{1}{8}\pi)$. [3]

(iv) Hence find the value of $\operatorname{cosec}^2(\frac{1}{8}\pi) + \operatorname{cosec}^2(\frac{3}{8}\pi)$, justifying your answer. [5]

(Total 12 marks)

5.

(a) Using the identity $zz^* = |z|^2$, or otherwise, show that if w is any root of unity then

$$|w - 2|^2 = 5 - 2(w + w^*)$$

(3)

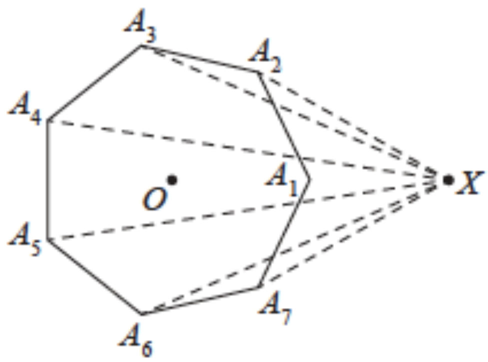

Figure 1

Figure 1 shows a regular heptagon $A_1A_2A_3A_4A_5A_6A_7$ whose vertices all lie on the unit circle with centre at the origin O and A_1 at (1, 0). The point X lies in the same plane as the heptagon and has coordinates (2, 0).

Using the result given in part (a),

(b) find $\sum_{i=1}^{7}(XA_i)^2$

(4)

(Total 7 marks)

6.

The series C and S are defined for $0 < \theta < \pi$ by

$$C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta,$$
$$S = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta.$$

(i) Show that $C + iS = \dfrac{e^{3i\theta} - e^{-3i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}\, e^{\frac{5}{2}i\theta}$. [4]

(ii) Deduce that $C = \sin 3\theta \cos \frac{5}{2}\theta \operatorname{cosec} \frac{1}{2}\theta$ and write down the corresponding expression for S. [4]

(iii) Hence find the values of θ, in the range $0 < \theta < \pi$, for which $C = S$. [4]

(Total 12 marks)