Topic Z4 Complex numbers (Post-TT A) [57] MARKSCHEME

1.

(a) $(z_1 z_2 =)\ 6\mathrm{e}^{\frac{5}{12}\pi\mathrm{i}}$	B1 B1	For modulus = 6 For argument = $\frac{5}{12}\pi$
$\left(\frac{z_1}{z_2} = \frac{2}{3}\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}} = \right) \frac{2}{3}\mathrm{e}^{\frac{23}{12}\pi\mathrm{i}}$	M1 A1 **4**	For subtracting arguments For correct answer
(b) $\left(w^{-5} =\right) 2^{-5}\operatorname{cis}\left(-\frac{5}{8}\pi\right)$	M1 A1	For use of de Moivre For $-\frac{5}{8}\pi$ seen or implied
$= \frac{1}{32}\left(\cos\frac{11}{8}\pi + \mathrm{i}\sin\frac{11}{8}\pi\right)$	A1 **3**	For correct answer (allow 2^{-5} and $\operatorname{cis}\frac{11}{8}\pi$)
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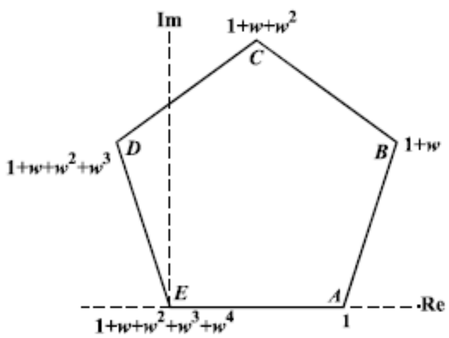
2.

				Allow $\operatorname{cis}\frac{k}{5}\pi$ and $\mathrm{e}^{\frac{k}{5}\pi\mathrm{i}}$ throughout
(i)	$w^2 = \cos\frac{4}{5}\pi + \mathrm{i}\sin\frac{4}{5}\pi$	B1		For correct value
	$w^3 = \cos\frac{6}{5}\pi + \mathrm{i}\sin\frac{6}{5}\pi$	B1		For correct value
	$w^* = \cos\frac{2}{5}\pi - \mathrm{i}\sin\frac{2}{5}\pi$	B1		For w^* seen or implied
	$= \cos\frac{8}{5}\pi + \mathrm{i}\sin\frac{8}{5}\pi$	B1	4	For correct value **SR** For exponential form with i missing, award B0 first time, allow others
(ii)		B1*		For $1+w$ in approximately correct position
		B1 (*dep)		For $AB \approx BC \approx CD$
		B1 (*dep)		For BC, CD equally inclined to Im axis
		B1	4	For E at the origin
				Allow points joined by arcs, or not joined Labels not essential
(iii)	$z^5 - 1 = 0$ *OR* $z^5 + z^4 + z^3 + z^2 + z = 0$	B1	1	For correct equation **AEF** (in any variable) Allow factorised forms using w, exp or trig
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3.

(i)	$2\cos\theta = \mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}$ $2^6\cos^6\theta = \mathrm{e}^{6i\theta} + 6\mathrm{e}^{4i\theta} + 15\mathrm{e}^{2i\theta} + 20 + 15\mathrm{e}^{-2i\theta} + 6\mathrm{e}^{-4i\theta} + \mathrm{e}^{-6i\theta}$	M1 A1	Expand $\left(e^{i\theta} + e^{-i\theta}\right)^6$	
	$2^6\cos^6\theta = \left(\mathrm{e}^{6i\theta} + \mathrm{e}^{-6i\theta}\right) +$ $\left(6\mathrm{e}^{4i\theta} + 6\mathrm{e}^{-4i\theta}\right) + \left(15\mathrm{e}^{2i\theta} + 15\mathrm{e}^{-2i\theta}\right) + 20$ $\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$	M1	for converting to multiple angles	
	⇒ result	A1 [4]	Complete argument including pairing up of e.g. terms in z^4 and z^{-4}	Must equate
(ii)	$\cos 6\theta + 6\cos 4\theta + 2\cos 2\theta = 3$ $\Rightarrow \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 3 + 13\cos 2\theta + 10$	M1*	Use result from (i)	
	$\Rightarrow 32\cos^6\theta = 13(1 + \cos 2\theta)$	A1	Oe simplified form	
	$\Rightarrow 32\cos^6\theta = 13(2\cos^2\theta)$	*M1dep	Use double angle identity	
	$\Rightarrow \cos\theta = 0$ or $\cos^4\theta = \frac{13}{16}$	A1		
	$\theta = \frac{1}{2}\pi, 0.319, 2.82$	A1 [5]		

4.

(i) $\cos 4\theta + i\sin 4\theta =$ $c^4 + 4ic^3 s - 6c^2 s^2 - 4ics^3 + s^4$ $\Rightarrow \sin 4\theta = 4c^3 s - 4cs^3$ and $\cos 4\theta = c^4 - 6c^2 s^2 + s^4$ $\Rightarrow \tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	M1 A1 M1 A1 **4**	For using de Moivre with $n = 4$ For both expressions For expressing $\frac{\sin 4\theta}{\cos 4\theta}$ in terms of c and s For simplifying to correct expression
(ii) $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$	B1 **1**	For inverting (i) and using $\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta = \frac{1}{\cot\theta}$. **AG**
(iii) $\cot 4\theta = 0$ Put $x = \cot^2\theta$ $\theta = \frac{1}{8}\pi \Rightarrow x^2 - 6x + 1 = 0$ *OR* $x^2 - 6x + 1 = 0 \Rightarrow \theta = \frac{1}{8}\pi$	B1 B1 B1 **3**	For putting $\cot 4\theta = 0$ (can be awarded in **(iv)** if not earned here) For putting $x = \cot^2\theta$ in the numerator of **(ii)** For deducing quadratic from **(ii)** and $\theta = \frac{1}{8}\pi$ *OR* For deducing $\theta = \frac{1}{8}\pi$ from **(ii)** and quadratic
(iv) $4\theta = \frac{3}{2}\pi$ *OR* $\frac{1}{2}(2n+1)\pi$ 2nd root is $x = \cot^2\left(\frac{3}{8}\pi\right)$ $\Rightarrow \cot^2\left(\frac{1}{8}\pi\right) + \cot^2\left(\frac{3}{8}\pi\right) = 6$ $\Rightarrow \operatorname{cosec}^2\left(\frac{1}{8}\pi\right) + \operatorname{cosec}^2\left(\frac{3}{8}\pi\right) = 8$	M1 A1 M1 M1 A1 **5** **13**	For attempting to find another value of θ For the other root of the quadratic For using sum of roots of quadratic For using $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ For correct value

5.

4(a)	$\lvert w-2\rvert^2 = (w-2)(w-2)^* = (w-2)(w^*-2)$	M1	1.1b
	$= ww^* - 2w - 2w^* + 4 = \lvert w\rvert^2 - 2(w + w^*) + 4$	M1	1.1b
	$= 1 + 4 - 2(w + w^*) = 5 - 2(w + w^*)$ since w is a root of unity so has modulus 1. *	A1*	2.1
		(3)	
Alt	$w = x + iy \Rightarrow \lvert w-2\rvert^2 = \lvert (x-2) + iy\rvert^2 = (x-2)^2 + y^2$	M1	1.1b
	$= x^2 - 4x + 4 + y^2 = x^2 + y^2 + 4 - 2(x + iy + x - iy)$	M1	1.1b
	$= 1 + 4 + 2(w + w^*)$ since $x^2 + y^2 = 1$ as w is a root of unity. *	A1*	2.1
		(3)	
(b)	$\sum_{i=1}^{7}(XA_i)^2 = \sum_{i=1}^{7}\lvert w_i - 2\rvert^2$ where w_i are the 7th roots of unity.	M1	3.1a
	$= \sum_{i=1}^{7}\left(5 - 2(w_i + w_i^*)\right) = \sum_{i=1}^{7}5 - 2\sum_{i=1}^{7}(w_i + w_i^*)$	M1	1.1b
	$\sum_{i=1}^{7}(w_i + w_i^*) = 0$ since roots of unity sum to zero.	B1	2.2a
	So $\sum_{i=1}^{7}(XA_i)^2 = 7 \times 5 = 35$	A1	1.1b
		(4)	

6.

(i) $C+\mathrm{i}S=1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+e^{4i\theta}+e^{5i\theta}$	M1		For using de Moivre, showing at least 3 terms
$=\dfrac{e^{6i\theta}-1}{e^{i\theta}-1}$	M1 A1		For recognising GP For correct GP sum
$=\dfrac{e^{3i\theta}-e^{-3i\theta}}{e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta}}\cdot\dfrac{e^{3i\theta}}{e^{\frac{1}{2}i\theta}}=\dfrac{e^{3i\theta}-e^{-3i\theta}}{e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta}}e^{\frac{5}{2}i\theta}$	A1	**4**	For obtaining correct expression **AG**
(ii) $C+\mathrm{i}S=\dfrac{2\mathrm{i}\sin 3\theta}{2\mathrm{i}\sin\frac{1}{2}\theta}\cdot e^{\frac{5}{2}i\theta}$	M1 A1		For expressing numerator and denominator in terms of sines For $k\sin 3\theta$ and $k\sin\frac{1}{2}\theta$
$\mathrm{Re}\Rightarrow C=\sin 3\theta\cos\frac{5}{2}\theta\operatorname{cosec}\frac{1}{2}\theta$	A1		For correct expression **AG**
$\mathrm{Im}\Rightarrow S=\sin 3\theta\sin\frac{5}{2}\theta\operatorname{cosec}\frac{1}{2}\theta$	B1	**4**	For correct expression
(iii) $C=S\ \Rightarrow\ \sin 3\theta=0,\ \tan\frac{5}{2}\theta=1$	M1		For either equation deduced **AEF** Ignore values outside $0<\theta<\pi$
$\theta=\frac{1}{3}\pi,\frac{2}{3}\pi$	A1		For both values correct and no extras
$\theta=\frac{1}{10}\pi,\frac{1}{2}\pi,\frac{9}{10}\pi$	A2	**4**	For all values correct and no extras. Allow A1 for any 1 value *OR* all correct with extras
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