Topic Z4 Complex numbers (Post-TT B) [53]

1.

It is given that $z = e^{i\theta}$, where $0 < \theta < 2\pi$, and $w = \dfrac{1+z}{1-z}$.

(i) Prove that $w = i\cot\frac{1}{2}\theta$. **[3]**

(ii) Sketch separate Argand diagrams to show the locus of z and the locus of w. You should show the direction in which each locus is described when θ increases in the interval $0 < \theta < 2\pi$. **[3]**

2.

Find the cube roots of $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, giving your answers in the form $\cos\theta + i\sin\theta$, where $0 \leqslant \theta < 2\pi$. **[4]**

3.

(i) Show that $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2\cos\phi)z + 1$. [1]

(ii) Write down the seven roots of the equation $z^7 = 1$ in the form $e^{i\theta}$ and show their positions in an Argand diagram. [4]

(iii) Hence express $z^7 - 1$ as the product of one real linear factor and three real quadratic factors. [5]

4.

(i) Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

(ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

(iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

5.

(i) By expressing $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, or otherwise, show that

$$\cos^2\theta\sin^4\theta = \tfrac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ [6]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \cos^2\theta\sin^4\theta\,d\theta.$$ [3]

6.

Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**