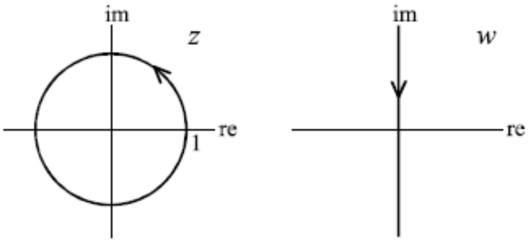
Topic Z4 Complex numbers (Post-TT B) [53] MARKSCHEME

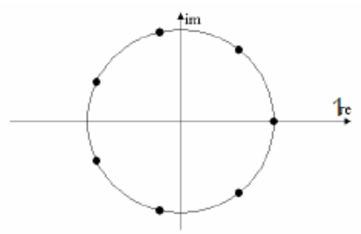
1.

(i)	METHOD 1 *EITHER* $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}=\dfrac{e^{-\frac{1}{2}i\theta}+e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}$	M1	*EITHER* For changing LHS terms to $e^{\pm\frac{1}{2}i\theta}$ *OR in reverse* For using $\cot\frac{1}{2}\theta=\dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$
	$=\dfrac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta}=i\cot\frac{1}{2}\theta$	M1	For either of $\begin{matrix}\cos\\ \sin\end{matrix}\frac{1}{2}\theta=\dfrac{e^{\frac{1}{2}i\theta}\pm e^{-\frac{1}{2}i\theta}}{(2)(i)}$ soi
	OR in reverse with similar working	A1 3	For fully correct proof to **AG** **SR** If factors of 2 or i are not clearly seen, award M1 M1 A0
(i)	METHOD 2 *EITHER* $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}\times\dfrac{1-e^{-i\theta}}{1-e^{-i\theta}}=\dfrac{e^{i\theta}-e^{-i\theta}}{2-\left(e^{i\theta}+e^{-i\theta}\right)}$ *OR* $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\times\dfrac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$	M1	For multiplying top and bottom by complex conjugate in exp or trig form
	$=\dfrac{2i\sin\theta}{2-2\cos\theta}=\dfrac{2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=i\cot\frac{1}{2}\theta$	M1 A1	For using both double angle formulae correctly For fully correct proof to **AG**
	METHOD 3 $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\dfrac{2\cos^2\frac{1}{2}\theta+2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$	M1	For using both double angle formulae correctly
	$=\dfrac{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}$	M1	For appropriate factorisation
	$=i\cot\frac{1}{2}\theta\dfrac{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}=i\cot\frac{1}{2}\theta$	A1	For fully correct proof to **AG**
	METHOD 4 $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\dfrac{1+\dfrac{1-t^2}{1+t^2}+i\dfrac{2t}{1+t^2}}{1-\dfrac{1-t^2}{1+t^2}-i\dfrac{2t}{1+t^2}}$	M1	For substituting both *t* formulae correctly
	$=\dfrac{2+2it}{2t^2-2it}=\dfrac{1}{t}\dfrac{1+it}{t-i}=\dfrac{i}{t}\dfrac{t-i}{t-i}=i\cot\frac{1}{2}\theta$	M1 A1	For appropriate factorisation For fully correct proof to **AG**
	METHOD 5 $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}\times\dfrac{1+e^{i\theta}}{1+e^{i\theta}}=\dfrac{1+2e^{i\theta}+e^{2i\theta}}{1-e^{2i\theta}}$ $=\dfrac{2+e^{i\theta}+e^{-i\theta}}{e^{-i\theta}-e^{i\theta}}$	M1	For multiplying top and bottom by $1+e^{i\theta}$ and attempting to divide by $e^{i\theta}$ *OR* multiplying top and bottom by $1+e^{-i\theta}$
	$=\dfrac{2(1+\cos\theta)}{-2i\sin\theta}=\dfrac{2\cos^2\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=\dfrac{\cos\frac{1}{2}\theta}{-i\sin\frac{1}{2}\theta}$	M1	For using both double angle formulae correctly
	$=i\cot\frac{1}{2}\theta$	A1 3	For fully correct proof to **AG**
(ii)		M1 A1 B1 3	For a circle centre *O* For indication of radius = 1 and anticlockwise arrow shown For locus of *w* shown as imaginary axis described downwards
		6	

2.

$\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}\mathrm{i}\right)^{\frac{1}{3}} = \left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)^{\frac{1}{3}}$	B1		For $\arg z = \frac{1}{6}\pi$ seen or implied
$= \cos\frac{1}{18}\pi + \mathrm{i}\sin\frac{1}{18}\pi$,	M1		For dividing $\arg z$ by 3
$\cos\frac{13}{18}\pi + \mathrm{i}\sin\frac{13}{18}\pi$,	A1		For any one correct root
$\cos\frac{25}{18}\pi + \mathrm{i}\sin\frac{25}{18}\pi$	A1	4	For 2 other roots and no more in range $0 \leqslant \theta < 2\pi$
	4		

3.

(i) $(z-\mathrm{e}^{\mathrm{i}\phi})(z-\mathrm{e}^{-\mathrm{i}\phi}) \equiv z^2 - (2)z\dfrac{(\mathrm{e}^{\mathrm{i}\phi}+\mathrm{e}^{-\mathrm{i}\phi})}{(2)} + 1$ $\equiv z^2 - (2\cos\phi)z + 1$	B1	1	For correct justification **AG**
(ii) $z = \mathrm{e}^{\frac{2}{7}k\pi\mathrm{i}}$ for $k = 0, 1, 2, 3, 4, 5, 6$ *OR* $0, \pm1, \pm2, \pm3$	B1 B1		For general form *OR* any one non-real root For other roots specified (k=0 may be seen in any form, eg 1, e^0, $\mathrm{e}^{2\pi\mathrm{i}}$) For answers in form $\cos\theta + \mathrm{i}\sin\theta$ allow maximum B1 B0
	B1 B1	4	For any 7 points equally spaced round unit circle (circumference need not be shown) For 1 point on $+^{\text{ve}}$ real axis, and other points in correct quadrants
(iii) $\left(z^7 - 1 =\right)(z-1)(z-\mathrm{e}^{\frac{2}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{4}{7}\pi\mathrm{i}})$ $(z-\mathrm{e}^{\frac{6}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-2}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-4}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-6}{7}\pi\mathrm{i}})$	M1		For using linear factors from **(ii)**, seen or implied
$= (z-\mathrm{e}^{\frac{2}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-2}{7}\pi\mathrm{i}}) \times (z-\mathrm{e}^{\frac{4}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-4}{7}\pi\mathrm{i}})$ $(z-\mathrm{e}^{\frac{6}{7}\pi\mathrm{i}})(z-\mathrm{e}^{\frac{-6}{7}\pi\mathrm{i}}) \times$	M1		For identifying at least one pair of complex conjugate factors
$\times (z-1)$	B1		For linear factor seen
$= (z^2 - (2\cos\frac{2}{7}\pi)z + 1) \times$	A1		For any one quadratic factor seen
$(z^2 - (2\cos\frac{4}{7}\pi)z + 1) \times (z^2 - (2\cos\frac{6}{7}\pi)z + 1) \times$ $\times (z-1)$	A1	5	For the other 2 quadratic factors and expression written as product of 4 factors
	10		

4.

(i)		$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$	B1	Or $\cos 5\theta = re\{(\cos\theta + \mathrm{i}\sin\theta)^5\}$	
		$= c^5 + 5\mathrm{i}c^4s - 10c^3s^2 - 10\mathrm{i}c^2s^3 + 5cs^4 + \mathrm{i}s^5$	M1		No more than 1 error, can be unsimplified
		$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$	M1	Take real parts	
		$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$	M1		
		$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$			
		$\cos 5\theta = 16c^5 - 20c^3 + 5c$	A1 [5]	AG	
(ii)		Multiplying by x gives $16x^5 - 20x^3 + 5x = 0$			Hence, so no marks for using quadratic at this stage.
		letting $x = \cos\alpha$ gives $\cos 5\alpha = 0$	M1		
		hence $5\alpha = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \frac{9}{2}\pi$	A1		
		$\alpha = \frac{1}{10}\pi, \frac{3}{10}\pi, \frac{5}{10}\pi, \frac{7}{10}\pi, \frac{9}{10}\pi$			
		$\cos\frac{5}{10}\pi = 0$ which is not a root	A1		
		so roots $x = \cos\frac{1}{10}\pi, \cos\frac{3}{10}\pi, \cos\frac{7}{10}\pi, \cos\frac{9}{10}\pi$	A1 [4]		
(iii)		$16x^4 - 20x^2 + 5 = 0 \Leftrightarrow x^2 = \dfrac{20 \pm \sqrt{80}}{32}$	B1		Can be gained if seen in (ii)
		cos decreases between 0 and π so $\cos\frac{1}{10}\pi$ is greatest root	M1		
		so $\cos\frac{1}{10}\pi = \sqrt{\dfrac{20+\sqrt{80}}{32}} = \sqrt{\dfrac{5+\sqrt{5}}{8}}$	A1 [3]	Dep on full marks in (ii)	

5.

4 **(i)** $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$, $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$	B1		For either expression, seen or implied z may be used for $e^{i\theta}$ throughout
$\Rightarrow \cos^2\theta\sin^4\theta = \frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2 \frac{1}{16}\left(e^{i\theta}-e^{-i\theta}\right)^4$			
$=\frac{1}{4}\left(e^{2i\theta}+2+e^{-2i\theta}\right).\frac{1}{16}\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{-4i\theta}\right)$	M1 A1 A1		For expanding terms For the 2 correct expansions SR Allow A1 A0 for $k\left(e^{2i\theta}+2+e^{-2i\theta}\right)\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{-4i\theta}\right)$, $k \neq \frac{1}{64}$
$=\frac{1}{64}\left(\left(e^{6i\theta}+e^{-6i\theta}\right)-2\left(e^{4i\theta}+e^{-4i\theta}\right)-\left(e^{2i\theta}+e^{-2i\theta}\right)+4\right)$	M1		For grouping terms and using multiple angles
$=\frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$ **AG**	A1	6	For answer obtained correctly
(ii) $\int_0^{\frac{1}{3}\pi}\cos^2\theta\sin^4\theta\,d\theta =$			
$=\frac{1}{32}\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{1}{3}\pi}$	M1 A1		For integrating answer to **(i)** For all terms correct
$=\frac{1}{32}\left[0+\frac{1}{4}\sqrt{3}-\frac{1}{4}\sqrt{3}+\frac{2}{3}\pi-0\right]=\frac{1}{48}\pi$	A1	3	For correct answer
		9	

6.

(i)	(a)	$e^{i\theta}+e^{2i\theta}+\ldots+e^{10i\theta}=\dfrac{e^{i\theta}\left(\left(e^{i\theta}\right)^{10}-1\right)}{e^{i\theta}-1}$ $=\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta}}$ $=\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}$	M1 A1 M1 A1 [4]	Sum of a GP AG	
(i)	(b)	$\theta = 2n\pi \Rightarrow \text{sum} = 10$	B1 [1]		
(ii)		$\cos\theta+\cos 2\theta+\ldots+\cos 10\theta=\mathrm{Re}\left(\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}\right)$ $=\dfrac{\mathrm{Re}\left(-ie^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)\right)}{2\sin\left(\frac{1}{2}\theta\right)}=\dfrac{\mathrm{Re}\left(-ie^{\frac{21}{2}i\theta}+ie^{\frac{1}{2}i\theta}\right)}{2\sin\left(\frac{1}{2}\theta\right)}$	M1 M1	Take real parts Manipulate expression	Must at least make genuine progress in sorting real part of numerator, or in converting numerator to trig terms.
		$=\dfrac{\sin\left(\frac{21}{2}\theta\right)-\sin\left(\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}$ $=\dfrac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}-\dfrac{1}{2}$	A1 [3]	AG	
(iii)		$\cos\frac{1}{11}\pi+\cos\frac{2}{11}\pi+\ldots+\cos\frac{10}{11}\pi=\dfrac{\sin\left(\frac{21}{22}\pi\right)}{2\sin\left(\frac{1}{22}\pi\right)}-\dfrac{1}{2}$ But $\sin\frac{21}{22}\pi=\sin\left(\pi-\frac{21}{22}\pi\right)=\sin\frac{1}{22}\pi$ So RHS $=\frac{1}{2}-\frac{1}{2}=0$, so $\frac{1}{11}\pi$ is a root Using $\sin(2\pi+x)=\sin x$ gives $2\pi+\frac{1}{2}\theta=\frac{21}{2}\theta\Rightarrow\theta=\frac{1}{5}\pi$	M1 M1 A1 A1 [4]	 AG	For second M1, must convince that solution is exact and not simply from calculator.