Topic Z4 Complex numbers (Pre-TT A) [50] MARKSCHEME

1.

(i)	$(z=)\ 2, -2,\ 2\mathrm{i}, -2\mathrm{i}$	M1		For at least 2 roots of the form $k\{1,\ \mathrm{i}\}$ **AEF**
		A1	**2**	For correct values
(ii)	$\frac{w}{1-w} = 2, -2,\ 2\mathrm{i}, -2\mathrm{i}$	M1		For $\frac{w}{1-w} =$ any one solution from **(i)**
	$w = \frac{z}{1+z}$	M1		For attempting to solve for w, using any solution or in general
	$w = \frac{2}{3}, 2$	B1		For any one of the 4 solutions
		A1		For both real solutions
	$w = \frac{4}{5} \pm \frac{2}{5}\mathrm{i}$	A1	**5**	For both complex solutions **SR** Allow B1√ and one A1√ from $k \neq 2$
		7		

2.

(i) $(\cos 6\theta =)\ \mathrm{Re}(c + \mathrm{i}s)^6$	M1		For expanding (real part of) $(c+\mathrm{i}s)^6$ at least 4 terms and 1 evaluated binomial coefficient needed
$(\cos 6\theta =)\ c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	A1		For correct expansion
$(\cos 6\theta =)$ $c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$	M1		For using $s^2 = 1 - c^2$
$(\cos 6\theta =)\ 32c^6 - 48c^4 + 18c^2 - 1$	A1	**4**	For correct result **AG**
(ii) $64x^6 - 96x^4 + 36x^2 - 3 = 0 \Rightarrow \cos 6\theta = \frac{1}{2}$	M1		For obtaining a numerical value of $\cos 6\theta$
$\Rightarrow (\theta =)\ \frac{1}{18}\pi, \frac{5}{18}\pi, \frac{7}{18}\pi$ etc.	A1		For any correct solution of $\cos 6\theta = \frac{1}{2}$
$\cos 6\theta = \frac{1}{2}$ has multiple roots	M1		For stating or implying at least 2 values of θ
largest x requires smallest θ $\Rightarrow$ largest positive root is $\cos\frac{1}{18}\pi$	A1	**4**	For identifying $\cos\frac{1}{18}\pi$ **AEF** as the largest positive root from a list of 3 positive roots *OR* from general solution *OR* from consideration of the cosine function
	8		

3.

(i)		METHOD 1		z may be used for $\mathrm{e}^{\mathrm{i}\theta}$ throughout	
		$\sin^3\theta\cos^2\theta = \left(\frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}\right)^3\left(\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}\right)^2$	B1	For $\left(\frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}\right)$ *OR* $\left(\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}\right)$ soi	
		$= -\frac{1}{32\mathrm{i}}\left(z^3 - 3z + 3z^{-1} - z^{-3}\right)\left(z^2 + 2 + z^{-2}\right)$	M1	For expanding brackets (binomial theorem or otherwise)	
			M1 B1	For full expansion with 12 terms. For $-\frac{1}{32\mathrm{i}}$	two brackets expanded soi by alternate method
		$= -\frac{1}{32\mathrm{i}}\left(\left(z^5 - z^{-5}\right) - \left(z^3 - z^{-3}\right) - 2\left(z - z^{-1}\right)\right)$	M1	For grouping terms	Can be seen at any stage
		$= -\frac{1}{16}\left(\frac{z^5 - z^{-5}}{2\mathrm{i}} - \frac{z^3 - z^{-3}}{2\mathrm{i}} - 2\frac{z - z^{-1}}{2\mathrm{i}}\right)$		This step, oe, is needed for the final mark	oe includes replacing z^5-z^{-5} with 2isin5θ etc
		$= -\frac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta)$	A1	For simplification to **AG** WWW	
			[6]		

		METHOD 2 $\sin^3\theta\cos^2\theta = \sin^3\theta - \sin^5\theta$ $2i\sin\theta = z - \frac{1}{z}$ $-8i\sin^3\theta = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$ $= (z^3 - \frac{1}{z^3}) - (3z - \frac{3}{z})$ $= 2i\sin 3\theta - 6i\sin\theta$ $32i\sin^5\theta = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ $= (z^5 - \frac{1}{z^5}) - (5z^3 - \frac{5}{z^3}) + (10z - \frac{10}{z})$ $= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\sin^3\theta\cos^2\theta$ $= -\frac{1}{32i}(4(2is3\theta - 6is\theta) + (2is5\theta - 10is3\theta + 20is\theta))$ $= -\frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 4\sin 3\theta + 10\sin\theta - 12\sin\theta)$ $= -\frac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta)$	B1 M1 M1 B1 B1 A1	For RHS * For grouping terms For RHS of this line and line * above For $-\frac{1}{32i}$ For ag www	
(ii)		$\sin^3\theta\cos^2\theta = 0 \Rightarrow \sin\theta = 0$ *OR* $\cos\theta = 0$ $\Rightarrow \theta = r\pi$ *OR* $\theta = (2r+1)\frac{1}{2}\pi$ $\Rightarrow \theta = \frac{n\pi}{2}$	M1 A1 A1 [3]	For either equation Accept also Sinθ = +/-1 For either solution, **AEF** including a list of the first few For both of above solutions leading to general solution in form of **AG** where $k = 2$	Can be implied by the A mark plus at least $\sin^3\theta = 0$ or similar. At least 2 in list (and no wrong solution)

4.

5 (i) *EITHER* $z = \sqrt{8}\,\text{cis}(2k+1)\frac{\pi}{4},\ k = 0, 1, 2, 3$ *OR* $z = \sqrt{8}\,e^{(2k+1)\frac{\pi}{4}i},\ k = 0, 1, 2, 3$	B1 B1 **2**	For correct modulus **AEF** For correct arguments **AEF**
(ii) $z = 2\sqrt{2}\left\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right\}$ $z = 2 + 2i, -2 + 2i, -2 - 2i, 2 - 2i$ $(z-\alpha), (z-\beta), (z-\gamma), (z-\delta)$	B1 B1 B1 B1 √ **4**	For any of $\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$ For any one value of z correct For all values of z correct **AEFcartesian** (may be implied from symmetry or factors) f.t., where $\alpha, \beta, \gamma, \delta$ are answers above
(iii) *EITHER* $(z-(2+2i))(z-(2-2i))$ $\times(z-(-2+2i))(z-(-2-2i))$ $= (z^2 + 4z + 8)(z^2 - 4z + 8)$	M1 M1 A1	For combining factors from **(ii)** in pairs Use of complex conjugate pairs For correct answer
OR $z^4 + 64 = (z^2 + az + b)(z^2 + cz + d)$ $\Rightarrow a + c = 0, b + ac + d = 0, ad + bc = 0, bd = 64$ Obtain $(z^2 + 4z + 8)(z^2 - 4z + 8)$	M1 M1 A1 **3** 9	For equating coefficients For solving equations For correct answer

5.

$(C+\mathrm{i}S=)\ \int_0^{\frac{1}{2}\pi} e^{2x}(\cos 3x+\mathrm{i}\sin 3x)(\mathrm{d}x)$		
$\cos 3x+\mathrm{i}\sin 3x=e^{3\mathrm{i}x}$	B1	For using de Moivre, seen or implied
$\int_0^{\frac{1}{2}\pi} e^{(2+3\mathrm{i})x}(\mathrm{d}x)=\frac{1}{2+3\mathrm{i}}\left[e^{(2+3\mathrm{i})x}\right]_0^{\frac{1}{2}\pi}$	M1* A1	For writing as a single integral in exp form For correct integration (ignore limits)
$=\frac{2-3\mathrm{i}}{4+9}\left(e^{(2+3\mathrm{i})\frac{1}{2}\pi}-e^0\right)=\frac{2-3\mathrm{i}}{13}\left(-\mathrm{i}e^{\pi}-1\right)$	A1	For substituting limits correctly (unsimplified) (may be earned at any stage)
	M1 (dep*)	For multiplying by complex conjugate of 2+3i
$=\left\{\frac{1}{13}\left(-2-3e^{\pi}+\mathrm{i}(3-2e^{\pi}\right)\right\}$	M1 (dep*)	For equating real and/or imaginary parts
$C=-\frac{1}{13}\left(2+3e^{\pi}\right)$	A1	For correct expression **AG**
$S=\frac{1}{13}\left(3-2e^{\pi}\right)$	A1	For correct expression
	8	

6.

4	(i)		Diagram	B1		must have triangle where B is anticlockwise from A, looks isosceles, AOB < $\frac{\pi}{4}$, if axes labelled then must be correct
			$OB=\lvert ze^{\frac{1}{6}\pi\mathrm{i}}\rvert=\lvert z\rvert\lvert e^{\frac{1}{6}\pi\mathrm{i}}\rvert=\lvert z\rvert.1=\lvert z\rvert=OA$	M1		condone $OB=\lvert z\rvert=OA$
			So triangle is isosceles oe	A1	without contradictions	
			$\angle AOB=\frac{1}{6}\pi$	B1	or 30°	Can be just on diagram
				[4]		
4	(ii)		$w=(1+\mathrm{i})+\left((5+2\mathrm{i})-(1+\mathrm{i})\right)e^{\pm\frac{1}{6}\pi\mathrm{i}}$	M1	Rotation of CD	Condone omission of ± in M marks
				M1 A1	Translation of attempted CE	
			$w=\frac{1}{2}+2\sqrt{3}+\left(3+\frac{1}{2}\sqrt{3}\right)\mathrm{i}$	M1	converts $e^{\pm\frac{1}{6}\pi\mathrm{i}}$ into $a+bi$ form	
			or $\frac{3}{2}+2\sqrt{3}+\left(-1+\frac{1}{2}\sqrt{3}\right)\mathrm{i}$	A1		
				[5]		
			Alternative method: $CE=\begin{pmatrix}a\\b\end{pmatrix}, CD=\begin{pmatrix}4\\1\end{pmatrix}$. Now use $CE\cdot CD=17\cos(\pi/6)$ and $CE^2=17$			
			to obtain equations $4a+b=17\sqrt{3}/2$ and $a^2+b^2=17$ (or equivalent)	M1 A1	(for both).	
			Obtain 3-term quadratic in one variable and solve to get one correct value of a or b	M1	Quadratics are $a^2-4\sqrt{3}a+47/4=0$ and $b^2-\sqrt{3}b-13/4=0$	
			$(a,b)=(2\sqrt{3}\pm\frac{1}{2},\frac{1}{2}\sqrt{3}\mp 2)$	A1		
			Final answer	A1		
				[5]		