Topic Z4 Complex numbers (Pre-TT B) [54]

1.

Find the smallest value θ of for which

$$(\cos\theta + \mathrm{i}\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-\mathrm{i})\ \{\theta \in \mathbb{R} : \theta > 0\}$$

[4 marks]

(Total 4 marks)

2.

(a) Show that $(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta}) = \frac{1}{16}(17-8\cos 2\theta)$

[3 marks]

(b) Given that the series $e^{2i\theta} + \frac{1}{4}e^{4i\theta} + \frac{1}{16}e^{6i\theta} + \frac{1}{64}e^{8i\theta} + ...$ has a sum to infinity, express this sum to infinity in terms of $e^{2i\theta}$

[2 marks]

(c) Hence show that $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \cos 2n\theta = \frac{16\cos 2\theta - 4}{17 - 8\cos 2\theta}$

[4 marks]

(d) Deduce a similar expression for $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \sin 2n\theta$

[1 mark]

(Total 10 marks)

3.

(i) Solve the equation $z^5 = 1$, giving your answers in polar form. [2]

(ii) Hence, by considering the equation $(z+1)^5 = z^5$, show that the roots of

$$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$$

can be expressed in the form $\frac{1}{e^{i\theta} - 1}$, stating the values of θ. [5]

(Total 7 marks)

4.

(i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1). \quad \textbf{[5]}$$

(iii) Hence find the exact value of

$$\cos(\tfrac{1}{12}\pi)\cos(\tfrac{5}{12}\pi)\cos(\tfrac{7}{12}\pi)\cos(\tfrac{11}{12}\pi),$$

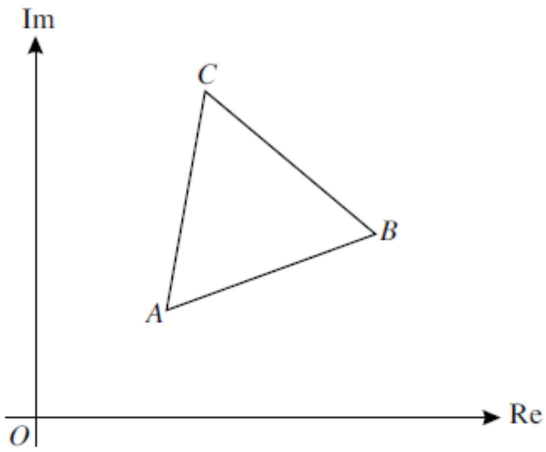
justifying your answer. **[5]**

(Total 13 marks)

5.

The cube roots of 1 are denoted by 1, ω and ω^2, where the imaginary part of ω is positive.

(i) Show that $1 + \omega + \omega^2 = 0$. **[2]**



In the diagram, ABC is an equilateral triangle, labelled anticlockwise. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.

(ii) State the geometrical effect of multiplication by ω and hence explain why $z_1 - z_3 = \omega(z_3 - z_2)$. **[4]**

(iii) Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$. **[2]**

(Total 10 marks)

6.

(i) By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10). \quad \textbf{[5]}$$

(ii) Replace θ by $(\frac{1}{2}\pi - \theta)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} (\sin^6\theta - \cos^6\theta)\,d\theta$. **[4]**

(Total 12 marks)