# Topic Z4 Complex numbers (Pre-TT B) [54] MARKSCHEME

1.

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| Uses De Moivre's theorem | AO3.1a | M1 | $(\cos\theta+\mathrm{i}\sin\theta)^5=\frac{1}{\sqrt{2}}(1-\mathrm{i})$ |
| Equates real and imaginary parts and obtains two equations | AO1.1a | A1 | $\Rightarrow\cos5\theta+\mathrm{i}\sin5\theta=\frac{1}{\sqrt{2}}(1-\mathrm{i})$ |
| Deduces that the smallest possible value of $5\theta$ is $\frac{7\pi}{4}$<br>FT from 'their' equations provided M1 has been awarded | AO2.2a | A1F | $\cos5\theta=\frac{1}{\sqrt{2}}\quad\sin5\theta=-\frac{1}{\sqrt{2}}$<br>$(5\theta=)\frac{7\pi}{4}$ |
| Obtains the smallest possible value of $\theta$ from fully correct reasoning<br>FT from 'their' $5\theta$ provided M1 has been awarded | AO1.1b | A1F | $\theta=\frac{7\pi}{20}$ |
| **Total** | | **4** | |

2.

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| **(a)** | Commences an argument by correctly expanding brackets and simplifying final term to 1/16 | AO1.1a | M1 | $(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})$<br>$=1-\frac{1}{4}e^{2i\theta}-\frac{1}{4}e^{-2i\theta}+\frac{1}{16}$ |
| | Substitutes correctly for both $e^{2i\theta}$ and $e^{-2i\theta}$ in terms of $\cos 2\theta$ and $\sin 2\theta$ (seen anywhere in solution) | AO1.1b | B1 | $=\frac{17}{16}-\frac{1}{4}(\cos2\theta+i\sin2\theta)-\frac{1}{4}(\cos2\theta-i\sin2\theta)$<br>$=\frac{17}{16}-\frac{1}{2}\cos2\theta$<br>$=\frac{1}{16}(17-8\cos2\theta)$ |
| | Completes argument and reaches stated result by collecting terms and simplifying correctly, no errors in working seen **AG** | AO2.1 | R1 | **AG** |
| **(b)** | Identifies series as a geometric series and states first term and common ratio correctly | AO1.1b | A1 | Geometric series with first term $r=e^{2i\theta}$ and common ratio $a=\frac{1}{4}e^{2i\theta}$ |
| | States and uses sum to infinity formula correctly<br>FT incorrect values for first term and common ratio | AO1.1b | A1F | $S_\infty=\frac{a}{1-r}=\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}$ |

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| **(c)** | Deduces that the series in part **(c)** is related to the real part of the series in part **(b)** | AO2.2a | R1 | Series stated = real part of the series $e^{2i\theta}+\frac{1}{4}e^{4i\theta}+\frac{1}{16}e^{6i\theta}+\frac{1}{64}e^{8i\theta}+....$ |
| | Selects an appropriate method by using the result in part **(b)** and multiplying appropriately to realise the denominator | AO3.1a | M1 | Using result from previous part $\dfrac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}=\dfrac{e^{2i\theta}}{(1-\frac{1}{4}e^{2i\theta})}\times\dfrac{(1-\frac{1}{4}e^{-2i\theta})}{(1-\frac{1}{4}e^{-2i\theta})}$ |
| | Substitutes to obtain an expression with cosines and sines only – using part **(a)**<br><br>FT incorrect sum to infinity provided M1 has been awarded | AO1.1b | A1F | $=\dfrac{e^{2i\theta}-\frac{1}{4}}{(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})}$<br>$\dfrac{\cos 2\theta-\frac{1}{4}+i\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}$ |
| | Identifies the real part and correctly completes the argument to reach the stated result.<br><br>Only award for an error-free fully correct solution | AO2.1 | R1 | Real part = $\dfrac{\cos 2\theta-\frac{1}{4}}{\frac{1}{16}(17-8\cos 2\theta)}=\dfrac{16\cos 2\theta-4}{17-8\cos 2\theta}$ |
| **(d)** | Identifies the imaginary part and states the correct expression | AO2.2a | R1 | Required series = imaginary part of the given series hence<br>$\dfrac{\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}=\dfrac{16\sin 2\theta}{17-8\cos 2\theta}$ |
| | **Total** | | **10** | |

3.

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| **(i)** | | $1, e^{\frac{2}{5}\pi i}, e^{\frac{4}{5}\pi i}, e^{\frac{6}{5}\pi i}, e^{\frac{8}{5}\pi i}$ oe polar form | M1<br>A1<br>[2] | Attempt roots | e.g. gives roots in an incorrect form. |
| **(ii)** | | $z^5=(z+1)^5=z^5+5z^4+10z^3+10z^2+5z+1$<br>$\Leftrightarrow 5z^4+10z^3+10z^2+5z+1=0$<br>so $z+1=ze^{\frac{2k}{5}\pi i}$, $k=0,1,2,3,4$<br>$k=0$ no solution<br>$1=z\left(e^{\frac{2k}{5}\pi i}-1\right)$<br>$z=\dfrac{1}{e^{\frac{2k}{5}\pi i}-1}$, $k=1,2,3,4$ | M1<br>A1<br>M1<br>B1<br><br>A1<br>[5] | <br><br><br>soi<br><br>If B0, then give A1 ft for correct solution plus $k=0$ | |

4.

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| **(i)** | $\cos 6\theta = 0 \Rightarrow 6\theta = k \times \frac{1}{2}\pi$ | M1 | | For multiples of $\frac{1}{2}\pi$ seen or implied |
| | $\Rightarrow \theta = \frac{1}{12}\pi\{1, 3, 5, 7, 9, 11\}$ | A1 | | A1 for any 3 correct |
| | | A1 | **3** | A1 for the rest, and no extras in $0 < \theta < \pi$ |
| **(ii)** | METHOD 1 | | | |
| | $\text{Re}(c + \text{i}s)^6 = \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$ | M1 | | For expanding $(c + \text{i}s)^6$ at least 4 terms and 2 binomial coefficients needed |
| | | A1 | | For 4 correct terms |
| | $\cos 6\theta = c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$ | M1 | | For using $s^2 = 1 - c^2$ |
| | $\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$ | A1 | | For correct expression for $\cos 6\theta$ |
| | $\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$ | A1 | **5** | For correct result **AG** (may be written down from correct $\cos 6\theta$ ) |
| | METHOD 2 | | | |
| | $\text{Re}(c + \text{i}s)^3 = \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ | M1 | | For expanding $(c + \text{i}s)^3$ at least 2 terms and 1 binomial coefficient needed |
| | | A1 | | For 2 correct terms |
| | $\Rightarrow \cos 6\theta = \cos 2\theta\left(\cos^2 2\theta - 3\sin^2 2\theta\right)$ | M1 | | For replacing $\theta$ by $2\theta$ |
| | $\Rightarrow \cos 6\theta = \left(2\cos^2\theta - 1\right)\left(4\left(2\cos^2\theta - 1\right)^2 - 3\right)$ | A1 | | For correct expression in $\cos\theta$ (unsimplified) |
| | $\Rightarrow \cos 6\theta = \left(2c^2 - 1\right)\left(16c^4 - 16c^2 + 1\right)$ | A1 | | For correct result **AG** |
| **(iii)** | METHOD 1 | | | |
| | $\cos 6\theta = 0$ | M1 | | For putting $\cos 6\theta = 0$ |
| | $\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $16c^4 - 16c^2 + 1 = 0$ and $2c^2 - 1 = 0$ | A1 | | For association of roots with quartic and quadratic |
| | But $\theta = \frac{1}{4}\pi, \frac{3}{4}\pi$ satisfy $2c^2 - 1 = 0$ | B1 | | For correct association of roots with quadratic |
| | *EITHER* Product of 4 roots *OR* $c = \pm\frac{1}{2}\sqrt{2 \pm \sqrt{3}}$ | M1 | | For using product of 4 roots *OR* for solving quartic |
| | $\Rightarrow \cos\frac{1}{12}\pi\cos\frac{5}{12}\pi\cos\frac{7}{12}\pi\cos\frac{11}{12}\pi = \frac{1}{16}$ | A1 | **5** | For correct value (may follow A0 and B0) |
| | METHOD 2 | | | |
| | $\cos 6\theta = 0$ | M1 | | For putting $\cos 6\theta = 0$ |
| | $\Rightarrow$ 6 roots of $\cos 6\theta = 0$ satisfy $32c^6 - 48c^4 + 18c^2 - 1 = 0$ | A1 | | For association of roots with sextic |
| | Product of 6 roots $\Rightarrow$ | M1 | | For using product of 6 roots |
| | $\cos\frac{1}{12}\pi . \frac{1}{\sqrt{2}} . \cos\frac{5}{12}\pi\cos\frac{7}{12}\pi . \frac{-1}{\sqrt{2}} . \cos\frac{11}{12}\pi = -\frac{1}{32}$ | B1 | | For using $\cos\left\{\frac{3}{12}\pi, \frac{9}{12}\pi\right\} = \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\}$ |
| | $\cos\frac{1}{12}\pi\cos\frac{5}{12}\pi\cos\frac{7}{12}\pi\cos\frac{11}{12}\pi = \frac{1}{16}$ | A1 | | For correct value |
| | | **13** | | |

5.

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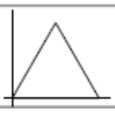
| (i) | *EITHER* $1+\omega+\omega^2$ $=$ sum of roots of $(z^3-1=0)=0$ | M1<br>A1 | 2 | For result shown by any correct method **AG** |
| | *OR* $\omega^3=1 \Rightarrow (\omega-1)(\omega^2+\omega+1)=0$<br>$\Rightarrow 1+\omega+\omega^2=0$ (for $\omega \ne 1$) | | | |
| | *OR* sum of G.P.<br>$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}\left(=\frac{0}{1-\omega}\right)=0$ | | | |
| | *OR*  shown on Argand diagram or explained in terms of vectors | | | |
| | *OR*<br>$1+\text{cis}\,\frac{2}{3}\pi+\text{cis}\,\frac{4}{3}\pi=1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right)=0$ | | | |
| (ii) | Multiplication by $\omega \Rightarrow$ rotation through $\frac{2}{3}\pi$ ↻ | B1 | | For correct interpretation of $\times$ by $\omega$<br>(allow 120° and omission of, or error in, ↻) |
| | $z_1-z_3=\overrightarrow{CA}$, $z_3-z_2=\overrightarrow{BC}$ | B1 | | For identification of vectors soi<br>(ignore direction errors) |
| | $\overrightarrow{BC}$ rotates through $\frac{2}{3}\pi$ to direction of $\overrightarrow{CA}$ | M1 | | For linking $BC$ and $CA$ by rotation of $\frac{2}{3}\pi$ *OR* $\omega$ |
| | $\Delta\,ABC$ has $BC=CA$, hence result | A1 | **4** | For stating equal magnitudes $\Rightarrow$ **AG** |
| (iii) | **(ii)** $\Rightarrow z_1+\omega z_2-(1+\omega)z_3=0$ | M1 | | For using $1+\omega+\omega^2=0$ in **(ii)** |
| | $1+\omega+\omega^2=0 \Rightarrow z_1+\omega z_2+\omega^2 z_3=0$ | A1 | 2 | For obtaining **AG** |
| | | | **8** | |

6.

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| (i) | $\sin\theta=\frac{1}{2\text{i}}\left(e^{\text{i}\theta}-e^{-\text{i}\theta}\right)$ | B1 | | $z$ may be used for $e^{\text{i}\theta}$ throughout<br>For expression for sinθ seen or implied |
| | | M1 | | For expanding $\left(e^{\text{i}\theta}-e^{-\text{i}\theta}\right)^6$<br>At least 4 terms and 3 binomial coefficients required. |
| | $\sin^6\theta=$<br>$-\frac{1}{64}\left(e^{6\text{i}\theta}-6e^{4\text{i}\theta}+15e^{2\text{i}\theta}-20+15e^{-2\text{i}\theta}-6e^{-4\text{i}\theta}+e^{-6\text{i}\theta}\right)$ | A1 | | For correct expansion. Allow $\frac{\pm(\text{i})}{64}(\ldots\ldots)$ |
| | $=-\frac{1}{64}(2\cos 6\theta-12\cos 4\theta+30\cos 2\theta-20)$ | M1 | | For grouping terms and using multiple angles |
| | $\sin^6\theta=-\frac{1}{32}(\cos 6\theta-6\cos 4\theta+15\cos 2\theta-10)$ | A1 | 5 | For answer obtained correctly **AG** |
| (ii) | $\cos^6\theta=$ *OR* $\sin^6\left(\frac{1}{2}\pi-\theta\right)=$ | M1 | | For substituting $\left(\frac{1}{2}\pi-\theta\right)$ for θ throughout |
| | $-\frac{1}{32}(\cos(3\pi-6\theta)-6\cos(2\pi-4\theta)+15\cos(\pi-2\theta)-10)$ | A1 | | For correct unsimplified expression |
| | $\cos^6\theta=\frac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)$ | A1 | 3 | For correct expression with $\cos n\theta$ terms **AEF** |
| (iii) | $\int_0^{\frac{1}{4}\pi}\frac{1}{32}(-2\cos 6\theta-30\cos 2\theta)\,d\theta$ | B1√ | | For correct integral. f.t. from $\sin^6\theta-\cos^6\theta$ |
| | $=-\frac{1}{16}\left[\frac{1}{6}\sin 6\theta+\frac{15}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$ | M1<br>A1√ | | For integrating $\cos n\theta$, $\sin n\theta$ *or* $e^{\text{i}n\theta}$<br>For correct integration. f.t. from integrand |
| | $=-\frac{11}{24}$ | A1 | 4 | For correct answer **WWW** |
| | | | **12** | |