Write yours and your teacher's name at the top of your answer sheets.

U6 Further Mathematics Mock

Paper 1 (Pure)
February 2020
2019-2020

Duration: 2 hour

Total number of marks: 96

Write your answers in the spaces provided.
Additional paper may be used if necessary.

You are permitted to use a scientific or graphical calculator in this paper.

Final answers should be given to a degree of accuracy appropriate to the context.

1.

Given that

$$f(x) = e^{2x} \cos x$$

(a) Show that

$$f''(x) = pf(x) + qf'(x)$$

where p and q are integers to be determined.

(5)

(b) Hence find the Maclaurin series for $f(x)$, in ascending powers of x, up to and including the term in x^5, giving each coefficient in its simplest form.

(3)

[8 marks]

2.

In this question you must show detailed reasoning.

The finite region R is enclosed by the curve with equation $y = \dfrac{8}{\sqrt{16+x^2}}$, the x-axis and the lines $x = 0$ and $x = 4$. Region R is rotated through 360° about the x-axis. Find the exact value of the volume generated. **[4]**

[4 marks]

3.

Solve the equation $z^3 = i$, giving your answers in the form $e^{i\theta}$, where $-\pi < \theta \leq \pi$

[4 marks]

[4 marks]

4.

The points $A(5, -4, 6)$ and $B(6, -6, 8)$ lie on the line L. The point C is $(15, -5, 9)$.

(a) D is the point on L that is closest to C.

Find the coordinates of D.

[6 marks]

(b) Hence find, in exact form, the shortest distance from C to L.

[2 marks]

[8 marks]

5.

The equation of a plane is $4x + 2y + z = 7$.
The point A has coordinates (9, 6, 1) and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[6]**

[6 marks]

6.

$$f(x) = \frac{1}{\sqrt{x^2 + 2x + 10}}$$

(a) Determine $\int f(x)\,dx$

(3)

(b) Hence show that the mean value of f(x) over the interval [2, 20] may be expressed in the form $a\ln(b + c\sqrt{2})$, where a, b and c are rational constants to be determined.

(3)

[6 marks]

7.

In this question you must show detailed reasoning.

(a) By writing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ show that

$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$. **[5]**

(b) Hence show that $\sin\frac{1}{8}\pi = \frac{1}{2}\sqrt[6]{20 - 14\sqrt{2}}$. **[3]**

[8 marks]

8.

(a) Show that

$$\cosh^3 x + \sinh^3 x = \frac{1}{4}e^{mx} + \frac{3}{4}e^{nx}$$

where m and n are integers.

[3 marks]

(b) Hence find $\cosh^6 x - \sinh^6 x$ in the form

$$\frac{a\cosh(kx) + b}{8}$$

where a, b and k are integers.

[5 marks]

[8 marks]

9.

You are given that $y = \tan^{-1}\sqrt{2x}$.

(a) Find $\frac{dy}{dx}$. **[2]**

(b) Show that $\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{(x + 2x^2)}dx = k\pi$ where k is a number to be determined in exact form. **[4]**

[6 marks]

10.

Find the general solution of the differential equation

$$x\frac{dy}{dx} - 2y = \frac{x^3}{\sqrt{4 - 2x - x^2}}$$

where $0 < x < \sqrt{5} - 1$

[7 marks]

[7 marks]

11.

An isolated island is populated by rabbits and foxes. At time t the number of rabbits is x and the number of foxes is y.

It is assumed that:

- The number of foxes increases at a rate proportional to the number of rabbits. When there are 200 rabbits the number of foxes is increasing at a rate of 20 foxes per unit period of time.
- If there were no foxes present, the number of rabbits would increase by 120% in a unit period of time.
- When both foxes and rabbits are present the foxes kill rabbits at a rate that is equal to 110% of the current number of foxes.
- At time $t = 0$, the number of foxes is 20 and the number of rabbits is 80.

(a) (i) Construct a mathematical model for the rate of change of the number of rabbits.

[9 marks]

(a) (ii) Use this model to show that the number of rabbits has doubled after approximately 0.7 units of time.

[1 mark]

(b) Suggest one way in which the model that you have used for the rate of change of the number of rabbits could be refined.

[1 mark]

[11 marks]

12.

The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

[8 marks]

13.

A particle, P, of mass M is released from rest and moves along a horizontal straight line through a point O. When P is at a displacement of x metres from O, moving with a speed v ms^{-1}, a force of magnitude $|8Mx|$ acts on the particle directed towards O. A resistive force, of magnitude $4Mv$, also acts on P.

(a) Initially P is held at rest at a displacement of 1 metre from O. Describe completely the motion of P after it is released.

Fully justify your answer.

[8 marks]

(b) It is decided to alter the resistive force so that the motion of P is critically damped.

Determine the magnitude of the resistive force that will produce critically damped motion.

[4 marks]

[12 marks]