# Finding the angle between a line and a plane

**Starter**

1. (Review of last lesson) State whether these lines are parallel to, intersect or lie in the plane $\mathbf{r}\,.\,(2\mathbf{i}-2\mathbf{j}-\mathbf{k})=6$. If possible, find the coordinates of the point of intersection.

(a) $\mathbf{r}=5\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (b) $\mathbf{r}=3\mathbf{i}-\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$

***Working:*** (a) $\mathbf{r}=5\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+2\mathbf{k})=\begin{pmatrix}5+2\lambda\\1+\lambda\\2+2\lambda\end{pmatrix}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix}$

***Substitute in plane:*** $2(5+2\lambda)-2(1+\lambda)-(2+2\lambda)=6$

$0\lambda=0$

The line lies in the plane.

(b) $\mathbf{r}=3\mathbf{i}-\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+3\mathbf{k})=\begin{pmatrix}3+2\lambda\\\lambda\\-1+3\lambda\end{pmatrix}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix}$

***Substitute in plane:*** $2(3+2\lambda)-2\lambda-(-1+3\lambda)=6$

$\lambda=1$

When $\lambda=1$, $\mathbf{r}=3\mathbf{i}-\mathbf{k}+2\mathbf{i}+\mathbf{j}+3\mathbf{k}=5\mathbf{i}+\mathbf{j}+2\mathbf{k}$.
The point of intersection is $(5,\ 1,\ 2)$.

2. Find the angle between the line $\mathbf{r}=\mathbf{i}+3\mathbf{j}+5\mathbf{k}+\lambda(2\mathbf{i}+4\mathbf{j}+\mathbf{k})$ and the plane $x-y+z=0$.

***Working:*** Find the angle between the direction vectors:

$(2\mathbf{i}+4\mathbf{j}+\mathbf{k})\,.\,(\mathbf{i}-\mathbf{j}+\mathbf{k})=-1$

$|2\mathbf{i}+4\mathbf{j}+\mathbf{k}|=\sqrt{2^2+4^2+1^2}=\sqrt{21}$

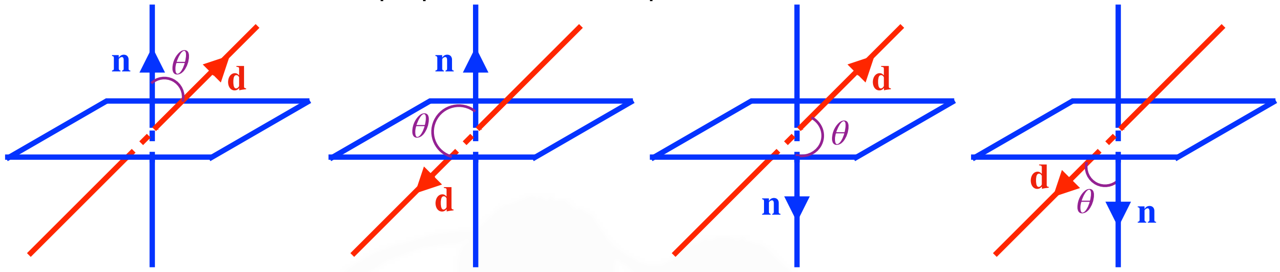
$|\mathbf{i}-\mathbf{j}+\mathbf{k}|=\sqrt{1^2+(-1)^2+1^2}=\sqrt{3}$

$\cos\theta=\dfrac{-1}{\sqrt{21}\sqrt{3}} \quad\Rightarrow\quad \theta\approx 97.24°$

Since lines and planes do not have a direction, give the acute angle.
The angle between the line and the plane is $\theta=7.24°$ (3 s.f.).

***Plane expressed in normal/Cartesian form***

***E.g. 1*** Here are the diagrams for the possibilities of directions of the direction vector parallel to the line, $\mathbf{d}$, and the vector perpendicular to the plane, $\mathbf{n}$.




In each case decide whether the angle calculated from $\mathbf{d}$ and $\mathbf{n}$ is acute or obtuse. Then state what is needed to be done to this angle in order to find the angle between the line and the plane. Generalise your findings.

***Working:*** 1st Diagram: angle between $\mathbf{d}$ and $\mathbf{n}$ is acute
Angle between line and plane: $90^\circ - \theta$

2nd Diagram: angle between $\mathbf{d}$ and $\mathbf{n}$ is obtuse
Angle between line and plane: $\theta - 90^\circ$

3rd Diagram: angle between $\mathbf{d}$ and $\mathbf{n}$ is obtuse
Angle between line and plane: $\theta - 90^\circ$

4th Diagram: angle between $\mathbf{d}$ and $\mathbf{n}$ is acute
Angle between line and plane: $90^\circ - \theta$

In general:
if the angle, $\theta$, between $\mathbf{d}$ and $\mathbf{n}$ is acute $\Rightarrow$ $90^\circ - \theta$
if the angle, $\theta$, between $\mathbf{d}$ and $\mathbf{n}$ is obtuse $\Rightarrow$ $\theta - 90^\circ$

***E.g. 2*** Find the angle between the line $\dfrac{x}{4} = 1 - y = \dfrac{3-z}{5}$ and the plane $\mathbf{r}\,.\,(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 0$.

***Working:*** Find the angle between the direction vectors:

$$(4\mathbf{i} - \mathbf{j} - 5\mathbf{k})\,.\,(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = -14$$

$$|4\mathbf{i} - \mathbf{j} - 5\mathbf{k}| = \sqrt{4^2 + (-1)^2 + (-5)^2} = \sqrt{42}$$

$$|\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$$

$$\cos\theta = \frac{-4}{\sqrt{42}\sqrt{21}} \quad \Rightarrow \quad \theta \approx 118.1^\circ$$

The angle between the line and the plane is $118.1^\circ - 90^\circ = 28.1^\circ$.

***E.g. 3*** Find the angle between the line $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k})$ and the plane given by:

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

***Working:*** Find the normal vector of the plane:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 5 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$$

Find the angle between $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$

$(2\mathbf{i} - \mathbf{j} - \mathbf{k}).(\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 11$

$|2\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$

$|\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}| = \sqrt{1^2 + (-2)^2 + (-7)^2} = \sqrt{54} = 3\sqrt{6}$

$$\cos\theta = \frac{11}{\sqrt{6} \times 3\sqrt{6}} \quad \Rightarrow \quad \theta \approx 52.3^\circ$$

Since the angle is acute, do $90^\circ - 52.3^\circ$

The angle between the line and the plane is $37.7^\circ$.

When the plane is expressed in ***parametric form***, i.e. $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u} + \mu\mathbf{v}$, ***find the normal vector*** by $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ and then use the method above.

**Video:** Angle between a line and a plane

Solutions to Starter and E.g.s

**Exercise**

p86 Ex 4C Qu 1i, 3, 5, 6, 8