Application of multiple angle formulae to polynomial equations

Starter

1. (Review of last lesson) (a) Find an expressions for $\cos 5\theta$ in terms of $\cos\theta$.

(b) Find the 5 values of θ, where $0 < \theta < \pi$, that satisfy $\cos 5\theta = 0$

(c) By putting the expression found in (a) equal to zero and solving, write the solutions of $\cos\theta$ in surd form.

(d) By writing your answers to (c) in descending values, find exact values for $\cos\dfrac{\pi}{10}$ and $\cos\dfrac{3\pi}{10}$.

Working: (a)

$\cos 5\theta \equiv \text{Re}\left[(c + \mathrm{i}s)^5\right]$

$(\cos\theta + \mathrm{i}\sin\theta)^5 = c^5 + 5\mathrm{i}c^4 s - 10c^3 s^2 - 10\mathrm{i}c^2 s^3 + 5cs^4 + \mathrm{i}s^5$

Equating real coefficients:

$$\begin{aligned}\cos 5\theta &\equiv c^5 - 10c^3 s^2 + 5cs^4 \\ &\equiv c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\ &\equiv c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 \\ &\equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

(b)

$\cos 5\theta = 0$

$$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \text{ and } \frac{9\pi}{2}$$

$$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10} = \frac{\pi}{2}, \frac{7\pi}{10} \text{ and } \frac{9\pi}{10}$$

(c)

From (a), $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$c(16c^4 - 20c^2 + 5) = 0$

$$c = 0 \quad \text{or} \quad c^2 = \frac{20 \pm \sqrt{(-20)^2 - 4\times 16\times 5}}{2\times 16}$$

$$c = 0 \quad \text{or} \quad c^2 = \frac{20 \pm \sqrt{80}}{32}$$

$$c = 0 \quad \text{or} \quad c^2 = \frac{5 \pm \sqrt{5}}{8}$$

$$c = 0 \quad \text{or} \quad c = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

(d)

In descending order the solutions are:

$$\sqrt{\frac{5+\sqrt{5}}{8}}, \sqrt{\frac{5-\sqrt{5}}{8}}, 0, -\sqrt{\frac{5-\sqrt{5}}{8}}, -\sqrt{\frac{5+\sqrt{5}}{8}}$$

Hence $\cos\dfrac{\pi}{10} = \sqrt{\dfrac{5+\sqrt{5}}{8}}$ and $\cos\dfrac{3\pi}{10} = \sqrt{\dfrac{5-\sqrt{5}}{8}}$

E.g. 1 (a) Solve $\cos 5\theta = 1$ for $0 \le \theta \le 2\pi$.

(b) State which solutions of $\cos 5\theta = 1$ are equal and hence state the number of distinct solutions of $\cos 5\theta = 1$

(b) By assuming that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, show that $\cos 5\theta = 1$ becomes $(c-1)(4c^2+2c-1)^2 = 0$.

(d) Hence find exact values for $\cos\dfrac{2\pi}{5}$ and $\cos\dfrac{4\pi}{5}$.

Hint: Are any of the solutions $\cos 5\theta = 1$ equal?

Working:

(a) $\cos 5\theta = 1$

$5\theta = 2\pi, 4\pi, 6\pi, 8\pi$ and 10π

$\theta = \dfrac{2\pi}{5}, \dfrac{4\pi}{5}, \dfrac{6\pi}{5}, \dfrac{8\pi}{5}$ and 2π

(b) $\cos\dfrac{2\pi}{5} = \cos\dfrac{8\pi}{5}$ and $\cos\dfrac{4\pi}{5} = \cos\dfrac{6\pi}{5}$

Hence there are three solutions (including 2π).

(c) $\cos 5\theta = 1 \quad\Rightarrow\quad 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 1$

$16c^5 - 20c^3 + 5c - 1 = 0$

By inspection $c = 1$ is a solution.

$16c^5 - 20c^3 + 5c - 1 = (c-1)(16c^4 + pc^3 + qc^2 + rc + 1)$

Equating coefficients:

c^4: $0 = -16 + p \quad\Rightarrow\quad p = 16$

c^3: $-20 = -p + q \quad\Rightarrow\quad q = -4$

c^2: $0 = -q + r \quad\Rightarrow\quad r = -4$

$16c^5 - 20c^3 + 5c - 1 = (c-1)(16c^4 + 16c^3 - 4c^2 - 4c + 1)$

Since there are only three solutions, $16c^4 + 16c^3 - 4c^2 - 4c + 1$ must have two pairs of repeated roots.

$16c^4 + 16c^3 - 4c^2 - 4c + 1 = (4c^2 + 2c - 1)^2$

$= (c-1)(4c^2+2c-1)^2$

Hence $\cos 5\theta = 1$ becomes $(c-1)(4c^2+2c-1)^2 = 0$

(d) Soving $4c^2 + 2c - 1 = 0$: $c = \dfrac{-2 \pm \sqrt{2^2 - 4\times 4\times(-1)}}{2}$

$c = \dfrac{-1\pm\sqrt{5}}{4}$

Since $\cos\dfrac{2\pi}{5} > 0$ and $\cos\dfrac{4\pi}{5} < 0$, $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$ and $\cos\dfrac{4\pi}{5} = \dfrac{-1-\sqrt{5}}{4}$.

E.g. 2 Given that $\sin 8\theta = 8\cos\theta\sin\theta(1 - 10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta)$ find an equation whose roots are $\pm\sin\frac{\pi}{8}$ and $\pm\sin\frac{3\pi}{8}$. Hence find an expression for $\sin\frac{\pi}{8}$ in surd form.

Working: Solve $\sin 8\theta = 0$: $8\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ and $\pm 4\pi$

$$\theta = 0, \pm\frac{\pi}{8}, \pm\frac{\pi}{4}, \pm\frac{3\pi}{8} \text{ and } \pm\frac{\pi}{2}$$

Consider $8\cos\theta\sin\theta(1 - 10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta) = 0$

The solution $\theta = 0^c$ comes from $\sin\theta = 0$.

The solutions $\theta = \pm\frac{\pi}{2}$ come from $\cos\theta = 0$.

Hence $16\sin^6\theta - 24\sin^4\theta + 10\sin^2\theta - 1 = 0$

To find an equation with roots $\pm\sin\frac{\pi}{8}$ and $\pm\sin\frac{3\pi}{8}$, the roots $\pm\sin\frac{\pi}{4}$ need to be removed

$$\left(s - \sin\frac{\pi}{4}\right)\left(s + \sin\frac{\pi}{4}\right) = \left(s - \frac{\sqrt{2}}{2}\right)\left(s + \frac{\sqrt{2}}{2}\right) = s^2 - \frac{1}{2}$$

Take $2s^2 - 1$ out as a factor:

$16s^6 - 24s^4 + 10s^2 - 1 = (2s^2 - 1)(8s^4 + ps^3 + qs^2 + rs + 1)$

Equating coefficients:

s^5: $0 = 2p \Rightarrow p = 0$

s^4: $-24 = -8 + 2q \Rightarrow q = -8$

s^3: $0 = 2r \Rightarrow r = 0$

$16s^6 - 24s^4 + 10s^2 - 1 = (2s^2 - 1)(8s^4 - 8s^2 + 1)$

The equation with $\pm\sin\frac{\pi}{8}$ and $\pm\sin\frac{3\pi}{8}$ as roots is $8s^4 - 8s^2 + 1 = 0$.

Solve $8s^4 - 8s^2 + 1 = 0$:

$$s^2 = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 8 \times 1}}{2 \times 8}$$

$$s^2 = \frac{8 \pm 4\sqrt{2}}{16} = \frac{2 \pm \sqrt{2}}{4}$$

$$s = \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2}$$

Since $\sin\frac{\pi}{8} < \sin\frac{3\pi}{8}$, $\sin\frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$.

E.g. 3 Given that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan\theta^3}{1 - 6\tan\theta^2 + \tan\theta^4}$,find the roots of $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.
Give your answers exactly.

Working: $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0 \quad \Rightarrow \quad t^4 - 6t^2 + 1 = 4t - 4t^3$

$$\Rightarrow \quad 1 = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

Letting $t = \tan\theta \quad \Rightarrow \quad \tan 4\theta = 1$

$$\Rightarrow \quad 4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}, \frac{29\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$$

But $\tan\dfrac{\pi}{16} = \tan\dfrac{17\pi}{16}$ etc.

So the solutions to the polynomial are:

$t = \tan\dfrac{\pi}{16},\ \tan\dfrac{5\pi}{16},\ \tan\dfrac{9\pi}{16},\ \tan\dfrac{13\pi}{16}$

$t = 0.199,\ 1.50,\ -5.03,\ -0.668$ (3 s.f.)

No Video:

Solutions to Starter and E.g.s

Exercise
p60 3B Qu 1-6