Area enclosed by a curve

Starter

1. **(Review of previous material)** Find the exact value of $\int_0^{\frac{\pi}{4}} (1 + 2\cos\theta)^2 d\theta$.

Working:

$$\int_0^{\frac{\pi}{4}} (1 + 2\cos\theta)^2 d\theta = \int_0^{\frac{\pi}{4}} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$
$$= \int_0^{\frac{\pi}{4}} (1 + 4\cos\theta + 2 + 2\cos 2\theta) d\theta$$
$$= \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{4}}$$
$$= \frac{3\pi}{4} + 2\sqrt{2} + 1$$

E.g. 1 A curve has equation $r = 3 + 3\sin\theta$. Find the area enclosed by the curve.

Working:

$$\text{Area enclosed} = \int_0^{2\pi} \frac{1}{2}(3 + 3\sin\theta)^2 d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (9 + 18\sin\theta + 9\sin^2\theta) d\theta$$
$$= \frac{9}{4}\int_0^{2\pi} (2 + 4\sin\theta + 1 - \cos 2\theta) d\theta$$
$$= \frac{9}{4}\Big[(3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta)\Big]_0^{2\pi}$$
$$= \frac{9}{4}\Big((6\pi - 4 - 0) - (0 - 4 - 0)\Big)$$
$$= \frac{27\pi}{2}$$

E.g. 2 A curve has equation $r = 5\cos 4\theta$. Find the area of one petal of the curve.

Working: $r \geq 0 \quad \Rightarrow \quad \cos 4\theta \geq 0$

Solving $\cos 4\theta = 0$: $\quad 4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \ldots$$

$r \geq 0$ for: $\quad -\frac{\pi}{8} \leq \theta \leq \frac{\pi}{8} \qquad \frac{3\pi}{8} \leq \theta \leq \frac{5\pi}{8}$

$$\frac{7\pi}{8} \leq \theta \leq \frac{9\pi}{8} \qquad \frac{11\pi}{8} \leq \theta \leq \frac{13\pi}{8}$$

Since there is symmetry of each loop:

Area of one loop $= 2 \times \int_0^{\frac{\pi}{8}} \frac{1}{2}(5\cos 4\theta)^2 d\theta$

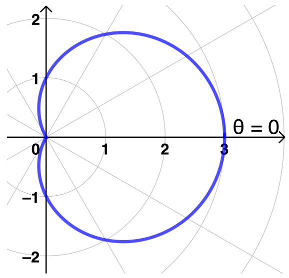
$$= 25\int_0^{\frac{\pi}{8}} \cos^2 4\theta d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{8}} (1 + \cos 8\theta) d\theta$$

$$= \frac{25}{2}\left[\theta + \frac{1}{8}\sin 8\theta\right]_0^{\frac{\pi}{8}}$$

$$= \frac{25\pi}{16}$$

E.g. 3 Find the area of the limaçon $r = 1 + 2\cos\theta$.



Working: $r \geq 0$: $\quad 1 + 2\cos\theta \geq 0 \quad \Rightarrow \quad \cos\theta \geq -\frac{1}{2}$

The curve only exists in the ranges $0 \leq \theta \leq \frac{2\pi}{3}$ and $\frac{4\pi}{3} \leq \theta \leq 2\pi$.

$$A = \int_0^{\frac{2\pi}{3}} \frac{1}{2} r^2 d\theta + \int_{\frac{4\pi}{3}}^{2\pi} \frac{1}{2} r^2 d\theta$$

$$= 2 \times \int_0^{\frac{2\pi}{3}} \frac{1}{2} r^2 d\theta$$

since there is symmetry

$$= \int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 d\theta$$

$$= \int_0^{\frac{2\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta) d\theta$$

$$= \int_0^{\frac{2\pi}{3}} (1 + 4\cos\theta + 2 + 2\cos 2\theta) d\theta$$

$$= \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_0^{\frac{2\pi}{3}}$$

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

Video: **Area bounded by a polar curve**

Solutions to Starter and E.g.s

Exercise
p211 9D Qu 1i, 2-9