Centres of mass by integration

Starter

1. (Review of last lesson) A hat consists of a hemispherical shell of radius 10 cm with a brim of outside radius 15 cm. Assuming the material is uniform thickness, find the distance of the centre of mass from the plane of the brim.

Working: Surface area of hemisphere $= 2\pi r^2 = 200\pi$

Surface area of brim $= \pi \times 15^2 - \pi \times 10^2 = 125\pi$

CoM of hemisphere $= \dfrac{1}{2} \times 10 = 5$ from the plane of the brim

↻ about the base of brim: $(200\pi + 125\pi)\bar{x} = 200\pi \times 5$

$$\bar{x} = \frac{1000}{325} = \frac{40}{13} \approx 3.0769$$

The centre of mass is 3.08 cm (3 s.f.) from the plane of the brim.

2. (Review of A2 FM Pure material) Find the volume of the solid formed when the curve $y = \cos x$ is rotated 2π about the $x-$axis between $x = 0$ and $x = \dfrac{\pi}{2}$.

Working:

$$\text{Volume} = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x \right) dx$$

$$= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \pi\left(\frac{1}{2}\frac{\pi}{2} + \frac{1}{4}\sin\pi \right) - \pi\Big(0 + 0\Big)$$

$$= \frac{\pi^2}{4}$$

E.g. 1 Find the coordinates of the centroid of the region between the curve $y = 4 - x^2$ and the positive $x-$ and $y-$axis.

Working: Solving $4 - x^2 = 0$ gives $x = \pm 2$ — we need to consider only the positive part.

$$A = \int_0^2 (4 - x^2)dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3}$$

$$A\bar{x} = \int_0^2 x(4 - x^2)dx = \int_0^2 (4x - x^3)dx = \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 4$$

$$A\bar{y} = \frac{1}{2}\int_0^2 (4 - x^2)^2 dx$$

$$= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)dx$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$$

$$= \frac{128}{15}$$

The coordinates of the centroid are $(0.75,\ 1.6)$

E.g. 2 A uniform lamina occupies the region bounded by the $x-$axis, the line $x = 2$ and the curve $y = 3x^2$ for $0 \leq x \leq 2$. Find the coordinates of the centre of mass of this lamina.

Working:

$$A = \int_0^2 3x^2 dx = \left[x^3\right]_0^2 = 8$$

$$A\bar{x} = \int_0^2 x \times 3x^2 dx = \int_0^2 3x^3 dx = \left[\frac{3}{4}x^4\right]_0^2 = 12$$

$$A\bar{y} = \frac{1}{2}\int_0^2 (3x^2)^2 dx = \frac{1}{2}\int_0^2 9x^4 dx = \frac{1}{2}\left[\frac{9}{5}x^5\right]_0^2 = \frac{144}{5}$$

$$\therefore \bar{x} = \frac{12}{8} = 1.5 \quad \text{and} \quad \bar{y} = \frac{144}{5 \times 8} = 3.6$$

The coordinates of the centre of mass are $(1.5,\ 3.6)$

E.g. 3 A machine component has the form of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$ for which $x \geq 0$ about the $x-$axis, the units being cm. Find the position of the centre of mass.

Working:

$$\int_0^9 \pi x y^2 dx = \pi \int_0^9 x(9-x)dx$$
$$= \pi \int_0^9 (9x - x^2)dx$$
$$= \pi \left[\frac{9}{2}x^2 - \frac{1}{3}x^3\right]_0^9$$
$$= 121.5\pi$$

$$\text{Volume} = \int_0^9 \pi(9-x)dx = \pi\left[(9x - \frac{1}{2}x^2)\right]_0^9 = 40.5\pi$$

So the $x-$coordinate of the centre of mass is $\dfrac{121.5\pi}{40.5\pi} = 3$ cm from the origin.

E.g. 4 The spike of a swordfish is modelled as the solid obtained by rotating the curve $y = 1 + 0.001x^{\frac{3}{2}}$ about the $x-$axis for $0 \leq x \leq 100$, the units being cm. How far from the end is the centre of mass of the spike?

Working:

$$\int_0^{100} \pi x y^2 dx = \pi \int_0^{100} x(1 + 0.001x^{\frac{3}{2}})^2 dx$$
$$= \pi \int_0^{100} (x + 0.002x^{\frac{5}{2}} + 0.000001x^4)dx$$
$$= \pi \left[\frac{1}{2}x^2 + \frac{1}{1750}x^{\frac{7}{2}} + \frac{1}{5000000}x^5\right]_0^{100}$$
$$= \pi\left(5000 + \frac{40000}{7} + 2000\right) - \pi(0)$$
$$= \frac{89000\pi}{7}$$

$$\int_0^{100} \pi y^2 dx = \pi \int_0^{100} (1 + 0.001x^{\frac{3}{2}})^2 dx$$
$$= \pi \int_0^{100} (1 + 0.002x^{\frac{3}{2}} + 0.000001x^3)dx$$
$$= \pi \left[x + \frac{1}{1250}x^{\frac{5}{2}} + \frac{1}{4000000}x^4\right]_0^{100}$$
$$= \pi\left(100 + 80 + 25\right) - \pi(0)$$
$$= 205\pi$$

The centre of mass is $\dfrac{\frac{89000\pi}{7}}{205\pi} = \dfrac{17800}{287} = 62.0$ cm (3 s.f.) from the end.

Video (password needed): Centre of mass of a lamina by integration
Video: Centre of mass by integration
Video: Centre of mass (solid of revolution)

Solutions to Starter and E.g.s

Exercise
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