Complex exponents (including Euler's formula)

Starter

1. **(Review of last lesson)** Show that $(-1+\mathrm{i})^{16}$ is real and that $\dfrac{1}{(-1+\mathrm{i})^6}$ is purely imaginary, giving the value of each.

Working:

$$(-1+\mathrm{i})^{16} = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)\right]^{16}$$
$$= 256\left(\cos 12\pi + \mathrm{i}\sin 12\pi\right)$$
$$= 256\left(\cos 0 + \mathrm{i}\sin 0\right)$$
$$= 256$$
$$\frac{1}{(-1+\mathrm{i})^6} = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)\right]^{-6}$$
$$= \frac{1}{8}\left[\cos\left(-\frac{9\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{2}\right)\right]$$
$$= \frac{1}{8}\left(\cos\frac{3\pi}{2} + \mathrm{i}\sin\frac{3\pi}{2}\right)$$
$$= -\frac{1}{8}\mathrm{i}$$

2. The standard Maclaurin series for e^x is $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots + \dfrac{x^r}{r!} + \ldots$.

(a) Find the Maclaurin series for $e^{\mathrm{i}\theta}$.

(b) By considering the Maclaurin series for $\cos\theta$ and $\sin\theta$, express $e^{\mathrm{i}\theta}$ in terms of $\cos\theta$ and $\sin\theta$.

Working: (a)

$$e^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \ldots$$
$$= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\theta^5}{5!} + \ldots$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$
$$e^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\theta^5}{5!} + \ldots$$
$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right)$$
$$= \cos\theta + \mathrm{i}\sin\theta$$

E.g. 1 State the value of $e^{\mathrm{i}\pi}$ — this is known as Euler's formula.

Working:

$$e^{\mathrm{i}\pi} = \cos\pi + \mathrm{i}\sin\pi = -1$$

i.e. $\text{irrational number}^{\text{imaginary number} \times \text{irrational number}} = -1$

E.g. 2 Express each of the complex number in the form $re^{i\theta}$:

(a) $1+\mathrm{i}$ (b) $\sqrt{3}-\mathrm{i}$

(c) $\dfrac{1+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ (d) $(1+\mathrm{i})(\sqrt{3}-\mathrm{i})$

Working: (a) $1+\mathrm{i}=\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$

(b) $\sqrt{3}-\mathrm{i}=2e^{-\mathrm{i}\frac{\pi}{6}}$ or $2e^{\mathrm{i}\frac{11\pi}{6}}$

(c) $\dfrac{1+\mathrm{i}}{\sqrt{3}-\mathrm{i}}=\dfrac{\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}}{2e^{-\mathrm{i}\frac{\pi}{6}}}=\dfrac{\sqrt{2}}{2}e^{\mathrm{i}\frac{5\pi}{12}}$

(d) $(1+\mathrm{i})(\sqrt{3}-\mathrm{i})=\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}\times 2e^{-\mathrm{i}\frac{\pi}{6}}=2\sqrt{2}e^{\mathrm{i}\frac{\pi}{12}}$

E.g. 3 Express in the form a + ib:

(a) $4e^{\mathrm{i}\frac{\pi}{3}}$ (b) $5e^{\mathrm{i}\pi}$ (c) $e^{-\mathrm{i}\frac{\pi}{2}}$

Working: (a) $4e^{\mathrm{i}\frac{\pi}{3}}=4\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)=4\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)=2+\mathrm{i}2\sqrt{3}$

(b) $5e^{\mathrm{i}\pi}=5(\cos\pi+\mathrm{i}\sin\pi)=5(-1+0\mathrm{i})=-5$

(c) $e^{-\mathrm{i}\frac{\pi}{2}}=\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)=0-\mathrm{i}=-\mathrm{i}$

E.g. 4 Find expressions for $e^{i\theta}+e^{-i\theta}$ and $e^{i\theta}-e^{-i\theta}$ in terms of $\cos\theta$ and/or $\sin\theta$

Working:

$$\begin{aligned}e^{i\theta}+e^{-i\theta}&=\cos\theta+\mathrm{i}\sin\theta+\cos(-\theta)+\mathrm{i}\sin(-\theta)\\&=\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta\\&=2\cos\theta\\e^{i\theta}-e^{-i\theta}&=\cos\theta+\mathrm{i}\sin\theta-\big(\cos(-\theta)+\mathrm{i}\sin(-\theta)\big)\\&=\cos\theta+\mathrm{i}\sin\theta-\cos\theta+\mathrm{i}\sin\theta\\&=2\mathrm{i}\sin\theta\end{aligned}$$

E.g. 5 Given that $z=2+3\mathrm{i}$, express e^z in the form $a+b\mathrm{i}$, giving your answer to 4 s.f..

Working: $e^z=e^{2+3\mathrm{i}}=e^2\times e^{3\mathrm{i}}=e^2(\cos 3^c+\mathrm{i}\sin 3^c)=-7.315+1.043\mathrm{i}$ (4 s.f.)

So if $z=x+\mathrm{i}y$, the definition of e^z is $e^z=e^{x+\mathrm{i}y}=e^x\times e^{\mathrm{i}y}=e^x(\cos y+\mathrm{i}\sin y)$.

Video: De Moivre's Theorem
Video: Exponential form (Euler's relation)

Solutions to Starter and E.g.s

Exercise
p33 2B Qu 1i, 2i, 3i, 4i, 5-12